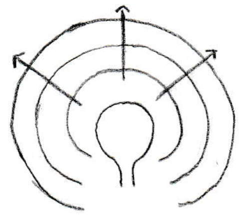
DRAWING #1: WAVES IN MULTIPLE DIMENSIONS

## WAVES IN MULTIPLE DIMENSIONS

SO FAR, WE HAVE ONLY BEEN CONSIDERING WAVES IN ONE DIMENSION.

MANY WAVES OF INTEREST, HOWEVER, ARE EMITTED BY SOURCES IN TWO OR THREE DIMENSIONS.

EXAMPLE: A LIGHT BULB



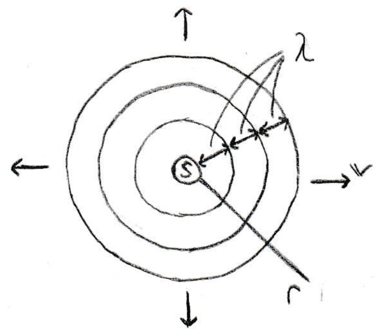
NOTE: THE LIGHT IS ALSO COMING OUT OF THE PAGE.

NOTE: THE FOLLOWING WILL DESCRIBE THREE DIMENSIONAL SPHERICAL WAVES; THE RESULTS CAN EASILY BE EXTENDED TO TWO DIMENSIONS.

DRAWING #2: SPHERICAL WAVES I

CONSIDER A GENERAL POINT SOURCE IN THREE DIMENSIONS THAT EMITS WAVES UNIFORMLY IN ALL DIRECTIONS --- i.e., THE WAVES WILL BE SPHERICALLY DISTRIBUTED.

TAKING A CROSS-SECTIONAL VIEW OF THIS SYSTEM AND DRAWING THE CRESTS OF THE EMITTED WAVE:




Ⓢ: POINT SOURCE

NOTE: THE LINES THAT LOCATE THE CRESTS ARE CALLED WAVE FRONTS.

--- i.e., THE CRESTS OF THE WAVE FORM CONCENTRIC SPHERICAL SHELLS SEPARATED BY THE WAVELENGTH $\lambda$.

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DUE TO THE SPHERICAL SYMMETRY OF THE PROBLEM, THERE IS ONLY ONE INDEPENDENT DIMENSION TO CONSIDER ($r$) AND WE CAN USE (MOST OF) THE MATHEMATICAL FORMULATION OF ONE-DIMENSIONAL WAVES.

IN PARTICULAR, A SOLUTION TO THE SPHERICALLY-SYMMETRIC THREE DIMENSIONAL WAVE EQUATION CAN BE WRITTEN (FOR AN OUTWARD-TRAVELLING WAVE):

$$\boxed{u(r,t) = A(r)\, e^{i(kr - \omega t + \phi)}}$$

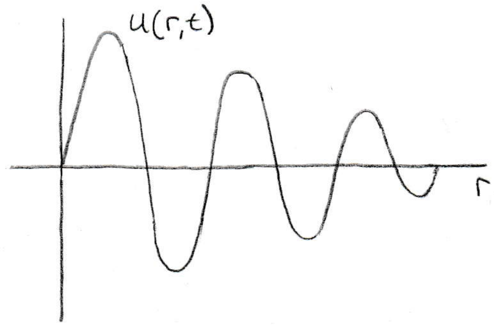
NOTE: ALL QUANTITIES HAVE BEEN DEFINED BEFORE, FOR ONE-DIMENSIONAL WAVES.

*NOTE: AN IMPORTANT DIFFERENCE BETWEEN THIS EQUATION AND THE ONE-DIMENSIONAL CASE IS THAT THE AMPLITUDE $A(r)$ IS A FUNCTION OF $r$:

WAVE ENERGY IS CONSERVED AS IT PROPAGATES. IN A SPHERICAL WAVE, FOR EXAMPLE, THE WAVE ENERGY IS SPREAD OUT OVER THE SPHERICAL SURFACE AREA. SINCE THE LATTER IS $4\pi r^2$, WE MIGHT EXPECT (CORRECTLY) THAT $A(r) \propto 1/r^2$. THIS IS CALLED SPHERICAL SPREADING LOSS.

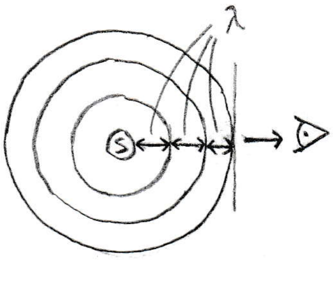
DRAWING #3: SPHERICAL WAVES 2

THE MATHEMATICAL FORM OF $u(r,t)$ (AND THE QUALITATIVE REMARKS BEHIND IT) SHOW THAT FOR AN DIRECTION $\hat{r}$, WE CAN CONSIDER AN OUTWARD PROPAGATING SINUSOIDAL WAVE WITH WAVELENGTH $\lambda$, FREQUENCY $\omega$, AND DECREASING AMPLITUDE:

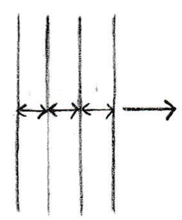


DRAWING #4: PLANE WAVES

IF A SPHERICAL WAVE IS OBSERVED VERY FAR FROM ITS SOURCE, THE WAVE FRONT IS A RELATIVELY SMALL AREA ELEMENT ON A RELATIVELY LARGE SPHERE; IF THE RADIUS OF THE SPHERE IS LARGE, THE CURVATURE OF THIS AREA WILL BE SMALL, AND THE WAVE FRONT WILL APPEAR TO BE A PLANE:




THE OBSERVED WAVE FRONTS WILL APPEAR TO BE IN PLANES:



SUCH WAVES ARE THEREFORE CALLED <u>PLANE WAVES</u>.

NOTE: PLANE WAVES DO NOT <u>NEED</u> TO ORIGINATE FROM A SPHERICALLY-SYMMETRIC EMITTING POINT SOURCE.

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BECAUSE THE DISPLACEMENT OF A PLANE WAVE DEPENDS ONLY ON THE ONE DIMENSION OF PROPAGATION (WE'LL CALL IT $x$), AND IS INDEPENDENT OF THE TRANSVERSE DIRECTIONS, WE CAN DESCRIBE SUCH WAVES WITH THE MATHEMATICAL FORMULATION OF ONE-DIMENSIONAL WAVES:

$$\boxed{u(x,t) = A\,e^{i(kx-\omega t+\phi)}}$$

NOTE: IF THE PLANE WAVE ORIGINATES AS PART OF A SPHERICAL WAVE, THEN A IS A FUNCTION OF $x$.

DRAWING #5: INTERFERENCE IN MULTIPLE DIMENSIONS 1

JUST AS IN ONE DIMENSION, WAVES IN MULTIPLE DIMENSIONS CAN SUPERPOSE, GIVING RISE TO CONSTRUCTIVE AND DESTRUCTIVE INTERFERENCE.

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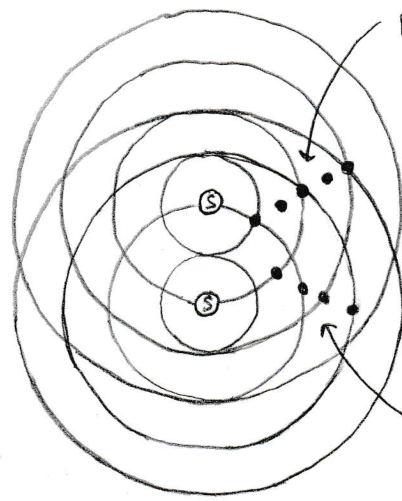
CONSIDER TWO (POINT) SOURCES EMITTING SPHERICAL (OR CIRCULAR WAVES) IN PHASE:




- CONSTRUCTIVE INTERFERENCE WILL OCCUR WHEN TWO CRESTS OR TROUGHS ALIGN
- DESTRUCTIVE " " CRESTS AND TROUGHS ALIGN

NOTE: SEVERAL POINTS OF CONSTRUCTIVE AND DESTRUCTIVE INTERFERENCE ARE SHOWN QUALITATIVELY BY THE WAVE FRONTS ABOVE.

IT IS IMPORTANT TO REALIZE THAT THE WAVE FRONTS ARE IN MOTION (TRAVELLING OUTWARD)...

... HOWEVER, THIS MOTION DOES NOT AFFECT THE POINTS OF CONSTRUCTIVE OR DESTRUCTIVE INTERFERENCE --- e.g., POINTS WHERE TWO CRESTS OVERLAP WILL BECOME POINTS WHERE TWO TROUGHS OVERLAP AND VICE VERSA, AND SO ON.

THE MATHEMATICAL DESCRIPTION OF INTERFERENCE IN MULTIPLE DIMENSIONS IS VERY SIMILAR TO THE ONE DIMENSIONAL CASE.

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CONSIDER SPHERICAL WAVES EMINATING FROM TWO DIFFERENT POINT SOURCES:

$$u_1(r_1,t) = A_1 e^{i(kr_1 - \omega t + \phi_1')}$$

$$u_2(r_2,t) = A_2 e^{i(kr_2 - \omega t + \phi_2')}$$

NOTE: HERE, AND IN WHAT FOLLOWS, WE WILL ASSUME THAT $A_1$ AND $A_2$ REMAIN CONSTANT OVER THE REGION OF INTEREST (KEEP IN MIND THAT THEY ARE DECREASING FUNCTIONS OF $r$).

RECALL THAT THE INTERFERENCE BETWEEN TWO WAVES DEPENDS ON (THE DIFFERENCE BETWEEN) THEIR PHASES:

$$\phi_1 = kr_1 - \omega t + \phi_1'$$

$$\phi_2 = kr_2 - \omega t + \phi_2'$$

$$\Delta\phi = \phi_2 - \phi_1$$

$$= kr_2 - \omega t + \phi_2' - (kr_1 - \omega t + \phi_1')$$

$$= kr_2 - kr_1 - \omega t + \omega t + \phi_2' - \phi_1'$$

$$= k(r_2 - r_1) + \phi_2' - \phi_1'$$

$$= k\Delta r + \Delta\phi'$$

WHERE: $\Delta r = r_2 - r_1$

$\Delta\phi' = \phi_2 - \phi_1$

THIS IS THE SAME RESULT AS BEFORE (FOR THE ONE DIMENSIONAL CASE), WITH $\Delta x$ REPLACED BY $\Delta r$.

WE CAN EASILY EXTEND OUR PRIOR RESULTS TO WRITE THE CONDITIONS FOR MAXIMUM CONSTRUCTIVE / DESTRUCTIVE INTERFERENCE:

$$\boxed{\Delta\phi = k\Delta r + \Delta\phi' = n2\pi \quad n = 0, 1, 2, \ldots}$$

(MAXIMUM CONSTRUCTIVE INTERENCE)

$$\boxed{\Delta\phi = k\Delta r + \Delta\phi' = \left(n + \tfrac{1}{2}\right)2\pi \quad n = 0, 1, 2, \ldots}$$

(MAXIMUM DESTRUCTIVE INTERFERENCE)