SUPERPOSITION OF WAVES

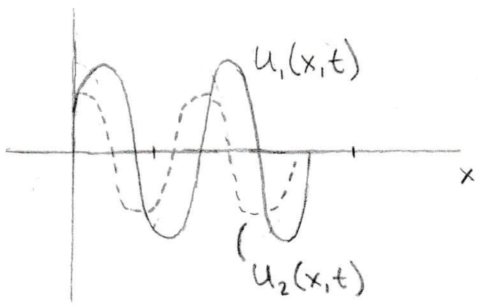
ONE OF THE KEY PROPERTIES OF WAVES IS THAT SUPERPOSITION OF SOLUTIONS TO THE WAVE EQUATION ALSO OBEYS THE WAVE EQUATION (THE PRINCIPLE OF SUPERPOSITION).

DRAWING #1.1: ASSUMPTION OF SINE WAVES

NOTE: WE HAVE FOUND THAT (i) WAVES OBEY THE SUPERPOSITION PRINCIPLE, AND (ii) ANY PERIODIC FUNCTION CAN BE REPRESENTED BY A SUM OF SINE FUNCTIONS. THEREFORE, WE CAN, FOR SIMPLICITY AND WITHOUT A LOSS OF GENERALITY, DERIVE ALL OF THE FOLLOWING RESULTS BY CONSIDERING ONLY TWO SINE WAVES.

DRAWING #2: INTERFERENCE I

CONSIDER TWO WAVES WITH THE SAME FREQUENCY AND WAVELENGTH TRAVELLING ALONG THE +X AXIS:



ASSUMING THAT THESE ARE SINE WAVES, THEY CAN BE REPRESENTED MATHEMATICALLY AS:

$$u_1(x,t) = A_1 e^{i(kx - \omega t + \phi_1)}$$

$$u_2(x,t) = A_2 e^{i(kx - \omega t + \phi_2)}$$

THE TWO WAVES HAVE DIFFERENT AMPLITUDES AND PHASES AT $x = 0$.

WE WOULD LIKE TO KNOW WHAT THE RESULTING WAVE LOOKS LIKE:

$$f(x,t) = u_1(x,t) + u_2(x,t)$$

INSERTING THE EXPRESSIONS FOR $u_1(x,t)$ AND $u_2(x,t)$:

$$\begin{aligned} f(x,t) &= A_1 e^{i(kx - \omega t + \phi_1)} + A_2 e^{i(kx - \omega t + \phi_2)} \\ &= A_1 e^{i\phi_1} e^{i(kx - \omega t)} + A_2 e^{i\phi_2} e^{i(kx - \omega t)} \\ &= \left(A_1 e^{i\phi_1} + A_2 e^{i\phi_2}\right) e^{i(kx - \omega t)} \end{aligned}$$

DEFINING:

$$\left(A_1 e^{i\phi_1} + A_2 e^{i\phi_2}\right) = A_f e^{i\phi_f}$$

WE CAN WRITE:

$$\begin{aligned} f(x,t) &= A_f e^{i\phi_f} e^{i(kx - \omega t)} \\ &= A_f e^{i(kx - \omega t + \phi_f)} \end{aligned}$$

DRAWING #3: INTERFERENCE 2

WE SEE THEREFORE THAT $f(x,t)$ IS A RIGHT-TRAVELLING WAVE WITH THE SAME FREQUENCY AND WAVELENGTH AS $u_1(x,t)$ AND $u_2(x,t)$, WITH AMPLITUDE A_f AND PHASE ϕ_f (BOTH OF WHICH WE CAN FIND IN TERMS OF A_1, A_2, ϕ_1, AND ϕ_2).

NOTE: WE SHOULD HAVE SUSPECTED THIS BY CONSIDERING THAT WAVES OBEY THE PRINCIPLE OF SUPERPOSITION.

INSIGHT INTO WHAT IS HAPPENING CAN BE OBTAINED BY DETERMINING WHAT A_f IS:

$$A_f e^{i\phi_f} = \left(A_1 e^{i\phi_1} + A_2 e^{i\phi_2}\right) \quad \text{(FROM BEFORE)}$$

$$A_f = \left(A_1 e^{i\phi_1} + A_2 e^{i\phi_2}\right) e^{-i\phi_f}$$

... AND FINDING ITS MAGNITUDE:

$$A_f^2 = A_f \cdot A_f^* \quad \text{(NOTE: THE * REPRESENTS THE COMPLEX CONJUGATE)}$$

$$= \left(A_1 e^{i\phi_1} + A_2 e^{i\phi_2}\right) e^{-i\phi_f} \times \left(A_1 e^{-i\phi_1} + A_2 e^{-i\phi_2}\right) e^{i\phi_f}$$

$$= \left[A_1^2 e^{i\phi_1} e^{-i\phi_1} + A_2 e^{i\phi_2} e^{-i\phi_2} + A_1 A_2 e^{-i\phi_1} e^{i\phi_2} + A_1 A_2 e^{i\phi_1} e^{-i\phi_2}\right] \times e^{-i\phi_f} e^{i\phi_f}$$

$$= \left[A_1^2 e^{i(\phi_1-\phi_1)} + A_2^2 e^{i(\phi_2-\phi_2)} + A_1 A_2 e^{i(\phi_2-\phi_1)} + A_1 A_2 e^{i(\phi_1-\phi_2)}\right] e^{i(\phi_f-\phi_f)}$$

$$= \left[A_1^2 e^0 + A_2^2 e^0 + A_1 A_2 \left(e^{i(\phi_2-\phi_1)} + e^{i(\phi_1-\phi_2)}\right)\right] e^0$$

DRAWING #4: INTERFERENCE 3

CONTINUING...

$$A_f \cdot A_f^* = A_1^2 + A_2^2 + A_1 A_2 \left(e^{i(\phi_2 - \phi_1)} + e^{i(\phi_1 - \phi_2)} \right)$$

$$\boxed{A_f \cdot A_f^* = A_1^2 + A_2^2 + 2A_1 A_2 \cos(\phi_2 - \phi_1)}$$

THE SUM OF TWO WAVES RESULTS IN A WAVE WITH AN INTENSITY THAT IS THE SUM OF: (i) THE INTENSITY OF THE FIRST WAVE ALONE, A_1^2; (ii) THE INTENSITY OF THE SECOND WAVE ALONE, A_2^2; AND (iii) A CORRECTION, $2A_1A_2\cos(\phi_2 - \phi_1)$ ...

... THIS CORRECTION IS CALLED THE <u>INTERFERENCE EFFECT</u>.

WE CAN USE THE PRIOR RESULT TO FIND THE PHASE OF $f(x,y)$:

WE KNOW THAT:

$$A_f e^{i\phi_f} = \left(A_1 e^{i\phi_1} + A_2 e^{i\phi_2} \right)$$

THEREFORE:

$$e^{i\phi_f} = \frac{1}{A_f} \left(A_1 e^{i\phi_1} + A_2 e^{i\phi_2} \right)$$

$$\cos\phi_f + i\sin\phi_f = \frac{1}{A_f} \left(A_1 \cos\phi_1 + A_1 i \sin\phi_1 + A_2 \cos\phi_2 + A_2 i \sin\phi_2 \right)$$

WHICH SEPARATES INTO REAL AND IMAGINARY PARTS:

$$\cos\phi_f = \frac{1}{A_f} \left(A_1 \cos\phi_1 + A_2 \cos\phi_2 \right)$$

$$\sin\phi_f = \frac{1}{A_f} \left(A_1 \sin\phi_1 + A_2 \sin\phi_2 \right)$$

(NOTE: i HAS BEEN DIVIDED OUT)

DRAWIN #4.1: INTERFERENCE 4

HENCE:

$$\phi_f = \cos^{-1}\left(\frac{1}{A_f}\left(A_1 \cos\phi_1 + A_2 \cos\phi_2\right)\right)$$

OR

$$\phi_f = \sin^{-1}\left(\frac{1}{A_f}\left(A_1 \sin\phi_1 + A_2 \sin\phi_2\right)\right)$$

NOTE: I'M NOT SURE THIS CAN BE SIMPLIFIED.

DRAWING #5: CONSTRUCTIVE AND DESTRUCTIVE INTERFERENCE!

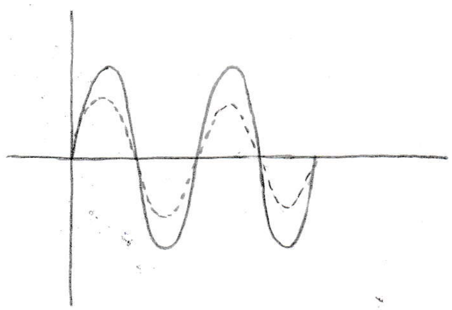
INTERFERENCE CAN BE POSITIVE OR NEGATIVE:

IF THE INTERFERENCE TERM IS POSITIVE, WE CALL IT CONSTRUCTIVE INTERFERENCE:

- THE AMPLITUDE OF THE SUPERPOSITION IS LARGER THAN EITHER OF THE TWO WAVES
- WAVES THAT ARE ALIGNED CREST-TO-CREST (AND TROUGH-TO-TROUGH):

$$\phi_2 - \phi_1 = n\, 2\pi \qquad n = 0, 1, 2, \ldots$$

ARE SAID TO BE IN PHASE.

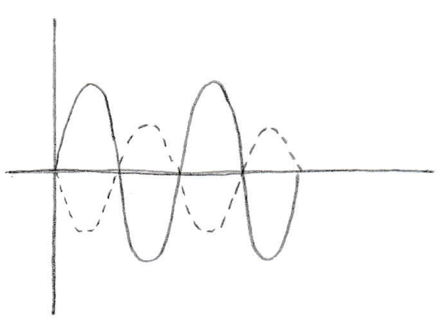


IF THE INTERFERENCE TERM IS NEGATIVE, IT IS CALLED DESTRUCTIVE INTERFERENCE:

- THE AMPLITUDE OF THE SUPERPOSITION IS SMALLER THAN THE SUM OF (THE SQUARES OF) THE TWO INDIVIDUAL WAVES
- WAVES THAT ARE ALIGNED CREST-TO-TROUGH:

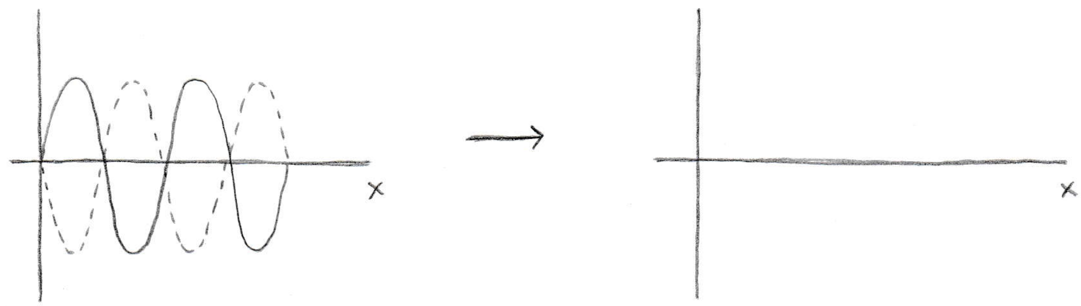
$$\phi_2 - \phi_1 = n\pi \qquad n = 1, 3, 5, \ldots$$

ARE SAID TO BE OUT OF PHASE.



DRAWING #6: CONSTRUCTIVE AND DESTRUCTIVE INTERFERENCE 2

- WAVES THAT ARE OUT OF PHASE AND HAVE THE SAME AMPLITUDE INTERFERE DESTRUCTIVELY, AND RESULT IN A SUPERPOSED WAVE WITH ZERO AMPLITUDE AT EVERY POINT ALONG THE AXIS: PERFECT DESTRUCTIVE INTERFERENCE.

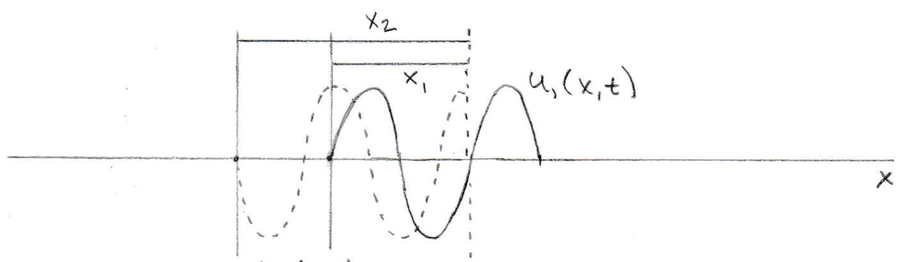


DRAWING #7 : INTERFERENCE IN SPACE I

SO FAR, WE HAVE BEEN CONSIDERING WAVES THAT ORIGINATE FROM THE SAME SOURCE (THE ORIGIN), POSSIBLY WITH DIFFERENT PHASES.

WE WILL NOW CONSIDER A MORE GENERAL CASE, WAVES ORIGINATING FROM DIFFERENT SOURCES.

CONSIDER TWO WAVES WITH THE SAME WAVELENGTH AND FREQUENCY TRAVELLING ALONG THE +X-AXIS, BUT ORIGINATING FROM DIFFERENT SOURCES:



WE WOULD LIKE TO KNOW WHAT THE RESULTING WAVE LOOKS LIKE:

$$f(x,t) = u_1(x_1,t) + u_2(x_2,t)$$

AT AN ARBITRARY POINT x, WHICH IS A DISTANCE OF x_1 FROM THE SOURCE OF WAVE 1 AND A DISTANCE OF x_2 FROM THE SOURCE OF WAVE 2.

ASSUMING THAT $u_1(x_1,t)$ AND $u_2(x_2,t)$ ARE SINE WAVES, THEY CAN BE REPRESENTED MATHEMATICALLY AS:

$$u_1(x_1,t) = A_1 e^{i(kx_1 - \omega t + \phi_1)}$$

$$u_2(x_2,t) = A_2 e^{i(kx_2 - \omega t + \phi_2)}$$

DRAWING #8: INTERFERENCE IN SPACE 2

WE HAVE SEEN EARLIER THAT THE INTERFERENCE BETWEEN TWO WAVES DEPENDS ON (THE DISTANCE BETWEEN) THEIR PHASES:

$$\phi_1 = kx_1 - \omega t + \phi_1'$$

$$\phi_2 = kx_2 - \omega t + \phi_2'$$

NOTE: THE UNPRIMED ϕ IS BEING USED AS THE (TOTAL) PHASE OF THE WAVE, WHILE A PRIME HAS BEEN ADDED TO THE PHASE CONSTANT.

THE DIFFERENCE BETWEEN THE TWO PHASES IS CALLED THE PHASE DIFFERENCE $\Delta\phi$:

$$\begin{aligned}
\Delta\phi &= \phi_2 - \phi_1 \\
&= (kx_2 - \omega t + \phi_2') - (kx_1 - \omega t + \phi_1') \\
&= kx_2 - kx_1 - \omega t + \omega t + \phi_2' - \phi_1' \\
&= k(x_2 - x_1) + \phi_2' - \phi_1' \\
&= k\,\Delta x + \Delta\phi'
\end{aligned}$$

WHERE: $\Delta x = x_2 - x_1$

$\Delta\phi' = \phi_2' - \phi_1'$

IT CAN BE SEEN THAT THERE ARE TWO CONTRIBUTIONS TO THE PHASE DIFFERENCE:

- $\Delta\phi'$: THE INHERENT PHASE DIFFERENCE BETWEEN THE TWO SOURCES
- $k\Delta x$; (PROPORTIONAL TO) THE DISTANCE BETWEEN THE TWO SOURCES (THE PATH-LENGTH DIFFERENCE) --- THE EXTRA DISTANCE TRAVELLED BY WAVE 2 RELATIVE TO WAVE 1 (TO THE POINT x)

RECALLING THAT WAVES (COMPLETELY) IN-PHASE:

$$\Delta\phi = n 2\pi \qquad n = 0, 1, 2, \ldots$$

WILL EXHIBIT (MAXIMUM) CONSTRUCTIVE INTERFERENCE, WE CAN WRITE:

$$\boxed{\Delta\phi = k\Delta x + \Delta\phi' = n 2\pi \qquad n = 0, 1, 2, \ldots \qquad \text{(MAXIMUM CONSTRUCTIVE INTERFERENCE)}}$$

NOTING THAT:

$$k = \frac{2\pi}{\lambda}$$

AND ASSUMING THAT $\Delta\phi' = 0$, WE CAN WRITE:

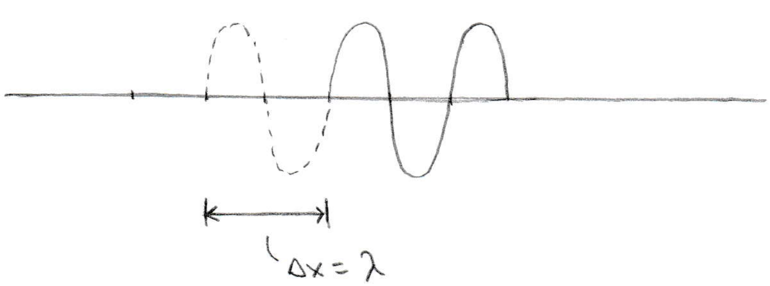
$$\frac{2\pi}{\lambda}\Delta x = n 2\pi$$

$$\frac{1}{\lambda}\Delta x = n$$

$$\Delta x = n\lambda$$

FROM WHICH WE SEE THAT TWO SOURCES PRODUCE MAXIMUM CONSTRUCTIVE INTERFERENCE WHEN THE PATH LENGTH DIFFERENCE IS AN INTEGER NUMBER OF WAVELENGTHS:

EXAMPLE:



DRAWING #10: INTERFERENCE IN SPACE 4

WAVES (COMPLETELY) OUT OF PHASE:

$$\Delta\phi = n\pi \qquad n = 1, 3, 5, \ldots$$

WILL EXHIBIT (MAXIMUM) DESTRUCTIVE INTERFERENCE.

WE CAN THEREFORE WRITE:

$$\boxed{\Delta\phi = k\Delta x + \Delta\phi' = \left(n + \tfrac{1}{2}\right)2\pi \qquad n = 0, 1, 2, \ldots}$$

(MAXIMUM DESTRUCTIVE INTERFERENCE)

NOTING THAT:

$$k = \frac{2\pi}{\lambda}$$

AND ASSUMING: $\Delta\phi' = 0$, WE CAN WRITE:

$$\frac{2\pi}{\lambda}\Delta x = \left(n + \frac{1}{2}\right)2\pi$$

$$\frac{1}{\lambda}\Delta x = \left(n + \frac{1}{2}\right)$$

$$\Delta x = \left(n + \frac{1}{2}\right)\lambda$$

FROM WHICH WE SEE THAT TWO SOURCES PRODUCE MAXIMUM DESTRUCTIVE INTERFERENCE WHEN THE PATH LENGTH DIFFERENCE IS A HALF-INTEGER NUMBER OF WAVELENGTHS.

EXAMPLE: