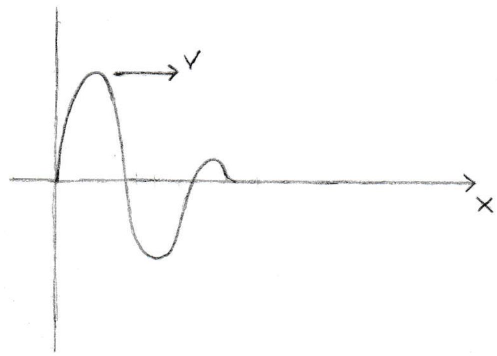
DRAWING #1: THE WAVE EQUATION

SUPPOSE THAT WE HAVE SOME GENERAL WAVE TRAVELLING ALONG THE X-AXIS WITH VELOCITY V:



WE CAN DESCRIBE THE WAVE BY A FUNCTION $u(x,t)$, WHICH GIVES DESCRIBES THE DISPLACEMENT OF THE WAVE AT ANY POSITION x AT ANY TIME t.

THE EQUATION DESCRIBING THE DISPLACMENT OF THE WAVE IS CALLED THE WAVE EQUATION:

$$\boxed{\frac{1}{v^2}\frac{\partial^2}{\partial t^2}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)}$$

NOTE THE USE OF PARTIAL DERIVATIVES.

NOTE: THIS EQUATION CAN BE DERIVED IN A VARIETY OF PHYSICAL SETTINGS (A STRING VIBRATING IN A TWO-DIMENSIONAL PLANE, WEIGHTS ALONG A ONE DIMENSIONAL AXIS CONNECTED BY SPRINGS, ETC.)...

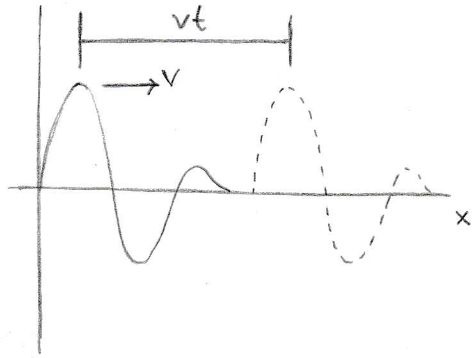
... IN FACT, THIS EQUATION IS THE GENERAL DIFFERENTIAL EQUATION THAT GOVERNS THE TRAVEL OF WAVES OF ALL TYPES.

(NOTE: WE'LL MAINLY BE INTERESTED IN ELECTROMAGNETIC WAVES)

NOTE: THIS EQUATION IS A SECOND-ORDER LINEAR PARTIAL DIFFERENTIAL EQUATION.

DRAWING #1.1: GENERAL SOLUTION TO THE WAVE EQUATION

A WAVE TRAVELLING TO THE RIGHT WITH VELOCITY V WILL HAVE UNDERGONE A DISPLACEMENT OF vt IN A TIME t:



THEREFORE, A GENERAL SOLUTION TO THE WAVE EQUATION CAN BE WRITTEN IN THE FORM:

$$u(x,t) = f(x - vt)$$

DRAWING #1.2: THE PRINCIPLE OF SUPERPOSITION

SUPPOSE THAT TWO SOLUTIONS TO THE WAVE EQUATION HAVE BEEN FOUND:

$$u_1(x,t)$$

$$u_2(x,t)$$

THIS MEANS THAT:

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}u_1(x,t) = \frac{\partial^2}{\partial x^2}u_1(x,t)$$

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}u_2(x,t) = \frac{\partial^2}{\partial x^2}u_2(x,t)$$

IF WE SUPERPOSE OUR TWO SOLUTIONS, WE HAVE:

$$f(x,t) = u_1(x,t) + u_2(x,t)$$

AND WE WISH TO VERIFY THAT THIS IS ALSO A SOLUTION TO THE WAVE EQUATION:

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}f(x,t) \stackrel{?}{=} \frac{\partial^2}{\partial x^2}f(x,t)$$

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\left(u_1(x,t) + u_2(x,t)\right) = \frac{\partial^2}{\partial x^2}\left(u_1(x,t) + u_2(x,t)\right)$$

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}u_1(x,t) + \frac{1}{v^2}\frac{\partial^2}{\partial t^2}u_2(x,t) =$$

$$\frac{\partial^2}{\partial x^2}u_1(x,t) + \frac{\partial}{\partial x^2}u_2(x,t)$$

WHICH SPLITS INTO TWO EQUATIONS:

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}u_1(x,t) = \frac{\partial^2}{\partial x^2}u_1(x,t)$$

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}u_2(x,t) = \frac{\partial^2}{\partial x^2}u_2(x,t)$$

DRAWING #1.3: THE PRINCIPLE OF SUPERPOSITION 2

WE KNOW THAT THE LATTER TWO EQUATIONS ARE SATISFIED...

HENCE: $f(x,t)$ IS INDEED SATISFIES THE WAVE EQUATION...

... AND WE HAVE VERIFIED THE PRINCIPLE OF SUPERPOSITION.

NOTE: THE PRINCIPLE OF SUPERPOSITION FOLLOWS FROM THE FACT THAT THE WAVE EQUATION IS LINEAR IN $u(x,t)$.

NOTE: IF WE CONSIDER THE ELECTROMAGNETIC FIELD AS A WAVE, THIS IS CONSISTENT WITH OUR FINDINGS THAT THE ELECTRIC AND MAGNETIC FIELDS OBEY THE PRINCIPLE OF SUPERPOSITION --- THE ABOVE IS MORE GENERAL, HOWEVER.

DRAWING #2: SOLUTIONS TO THE WAVE EQUATION I

ONE METHOD TO SOLVE ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS IS THE <u>SEPARATION OF VARIABLES</u>.

AS AN EXAMPLE, WE'LL APPLY THIS TO THE WAVE EQUATION:

ASSUME THAT THE SOLUTIONS TO THE WAVE EQUATION IN x AND t ARE SEPARATED:

$$u(x,t) = g(x)h(t)$$

AND SUBJECT TO HOMOGENEOUS BOUNDARY CONDITIONS:

$$u\Big|_{x=0} = u\Big|_{x=\lambda} = 0$$

(OR, MORE GENERALLY, THAT u IS PERIODIC, WITH A PERIOD OF λ)

(NOTE: THESE ASSUMPTIONS SHOULD BE CONSIDERED AS BEING MADE WITHOUT LOSS OF GENERALITY)

INSERTING THIS INTO THE WAVE EQUATION:

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) \qquad \text{(WAVE EQUATION)}$$

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\Big(g(x)h(t)\Big) = \frac{\partial^2}{\partial x^2}\Big(g(x)h(t)\Big) \qquad \text{(INSERT SEPARATED SOLUTION)}$$

$$\frac{1}{v^2}g(x)\frac{\partial^2}{\partial t^2}h(t) = h(t)\frac{\partial^2}{\partial x^2}g(x) \qquad \text{(PRODUCT RULE)}$$

$$\frac{1}{v^2}\frac{1}{h(t)}\frac{\partial^2}{\partial t^2}h(t) = \frac{1}{g(x)}\frac{\partial^2}{\partial x^2}g(x) \qquad \text{(REARRANGE)}$$

SINCE THE LEFT-HAND SIDE ON t, AND THE RIGHT-HAND SIDE ON x, BOTH SIDES ARE EQUAL TO SOME CONSTANT $-l$, WHERE $l > 0$.

NOTE: WITH HOMOGENEOUS BOUNDARY CONDITIONS, IT CAN BE SHOWN THAT $l > 0$ (OTHERWISE THE SOLUTION TO $g(x)$ OR $h(t)$ MUST BE IDENTICALLY ZERO)

(HENCE THE NEGATIVE SIGN ON l)

DRAWING #3: SOLUTIONS TO THE WAVE EQUATION 2

WE CAN THEREFORE WRITE:

$$\frac{1}{v^2}\frac{1}{h(t)}\frac{\partial^2}{\partial t^2}h(t) = -\ell$$

$$\frac{1}{g(x)}\frac{\partial^2}{\partial t^2}g(x) = -\ell$$

(NOTE: I HAVE KEPT THE DERIVATIVES IN PARTIAL DERIVATIVE NOTATION)

REARRANGING:

$$\frac{\partial^2}{\partial t^2}h(t) = -\ell v^2 h(t)$$

$$\frac{\partial^2}{\partial x^2}g(x) = -\ell g(x)$$

THESE EQUATIONS CAN BE SOLVED BY INSPECTION (AS WE DID FOR LC CIRCUITS):

WE KNOW THAT A FUNCTION WHOSE DERIVATIVE IS A CONSTANT MULTIPLE OF ITSELF IS THE EXPONENTIAL FUNCTION:

$$\frac{d}{dx}e^{rx} = re^{rx} \qquad \frac{d^2}{dx}e^{rx} = r^2e^{rx}$$

BY INSPECTION, THEREFORE:

$$g(x) = e^{rx}$$

$$h(t) = e^{st}$$

DRAWING #4: SOLUTIONS TO THE WAVE EQUATION 3

INSERTING THESE EXPRESSIONS INTO THE PRIOR EQUATION:

$$\frac{\partial^2}{\partial t^2}\left(e^{st}\right) = -\ell v^2 \left(e^{st}\right)$$

$$s^2\left(e^{st}\right) = -\ell v^2\left(e^{st}\right)$$

$$s^2 = -\ell v^2$$

$$s = \pm\sqrt{-\ell}\ v$$

$$s = \pm i\sqrt{\ell}\ v \qquad i = \sqrt{-1}$$

$$\frac{\partial^2}{\partial x^2}\left(e^{rx}\right) = -\ell\left(e^{rx}\right)$$

$$r^2\left(e^{rx}\right) = -\ell\left(e^{rx}\right)$$

$$r^2 = -\ell$$

$$r = \pm\sqrt{-\ell}$$

$$r = \pm i\sqrt{\ell}$$

DRAWING #5: SOLUTIONS TO THE WAVE EQUATION 4

WE CAN THEREFORE WRITE:

$$g(x) = e^{rx}$$
$$= e^{\pm i\sqrt{\ell}\, x}$$

$$h(t) = e^{st}$$
$$= e^{\pm i\sqrt{\ell}\, v t}$$

NOTE: THE ABOVE EQUATIONS STILL CONTAIN THE UNKNOWN CONSTANT ℓ ...

... WHICH CAN BE FOUND CONSIDERING THE BOUNDARY CONDITIONS.

SINCE u IS PERIODIC, WITH A PERIOD OF λ, IT MUST BE THE CASE THAT:

$$\sqrt{\ell} = n\frac{2\pi}{\lambda} \qquad n = 1, 2, 3, \ldots$$

EXAMPLE:

$$e^{\pm i n \frac{\pi}{\lambda} x} \stackrel{?}{=} e^{\pm i n \frac{2\pi}{\lambda}(x+\lambda)}$$
$$= e^{\pm\left(i n \frac{2\pi}{\lambda} x \,+\, i n \frac{2\pi}{\lambda} \lambda\right)}$$
$$= e^{\pm\left(i n \frac{2\pi}{\lambda} x \,+\, i n 2\pi\right)}$$
$$= e^{\pm i n \frac{2\pi}{\lambda} x}\, e^{\pm i n 2\pi}$$
$$= e^{\pm i n \frac{2\pi}{\lambda} x} \qquad \left(e^{\pm i n 2\pi} = 1 \text{ FOR } n = 1, 2, 3, \ldots\right)$$

WE CAN THEREFORE WRITE THE COMPLETE SOLUTION TO THE WAVE EQUATION AS:

$$u(x,t) = g(x)\,h(t)$$

$$= e^{\pm i\sqrt{\ell}\,x}\, e^{\pm i\sqrt{\ell}\,vt} \qquad \sqrt{\ell} = n\frac{2\pi}{\lambda}$$

$$= e^{\pm i\sqrt{\ell}\,(x+vt)}$$

OR

$$= e^{\pm i\sqrt{\ell}\,(x-vt)}$$

COMPARING THESE EXPRESSIONS WITH THE GENERAL FORMS OF RIGHT- AND LEFT-TRAVELLING WAVES THAT WE FOUND EARLIER, RESPECTIVELY:

$$f(x-vt)$$

$$f(x+vt)$$

WE SEE THAT THIS IS WHAT THEY IN FACT REPRESENT.

ADDITIONALLY, THE CHOICE OF $\pm$ DEPENDS ON OUR CONVENTION; WE'LL TAKE THE POSITIVE (+) SOLUTION.

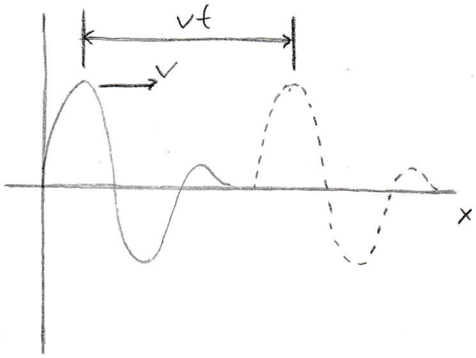
NOTE: FOR A FIXED n,

$$e^{iax}$$

REPRESENTS A SINUSOIDAL OSCILLATION ABOUT x.

DRAWING #7: A RIGHT-TRAVELLING WAVE

CONSIDER A WAVE TRAVELLING ALONG THE X-AXIS WITH VELOCITY v:



(NOTE: WE'VE SEEN THIS DRAWING BEFORE)

AFTER A TIME t, THE WAVE WILL HAVE MOVED A DISTANCE vt.

CONSIDER NOW (ONE OF) OUR SOLUTION(S) TO THE WAVE EQUATION:

$$u(x,t) = e^{i\sqrt{\ell}\,(x - vt)}$$

(NOTE: THIS SOLUTION REPRESENTS A SINUSOIDAL WAVE, WHILE THAT SHOWN IS A MORE GENERAL FORM --- QUALITATIVELY, THIS IS NOT A PROBLEM)

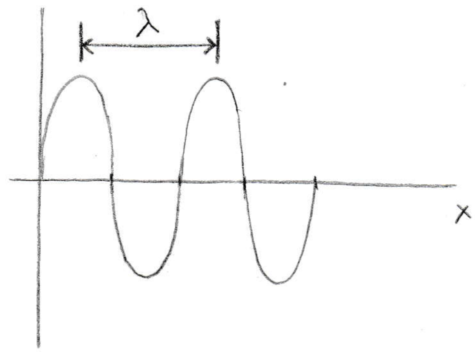
IT CAN BE SEEN THAT THIS SOLUTION IS INVARIANT TO THE DISPLACEMENT:

$$x - vt = \text{CONST.}$$

THIS SOLUTION THEREFORE IS CONSISTENT WITH OUR PICTURE OF A RIGHT-TRAVELLING WAVE.

DRAWING #8: ANGULAR WAVE NUMBER I

CONSIDER A SNAPSHOT OF OUR WAVE OVER ALL SPACE:



λ: WAVELENGTH OF A WAVE; DISTANCE BETWEEN REPETITIONS OF THE SHAPE OF THE WAVE (WAVE SHAPE)

FOR SIMPLICITY (AND WITHOUT LOSS OF GENERALITY), CONSIDER THE SNAPSHOT TO BE AT $t=0$.

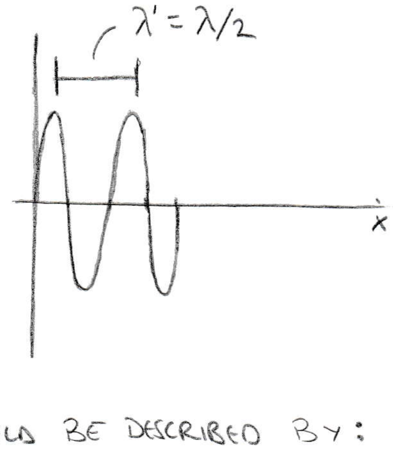
(ONE OF) OUR SOLUTIONS TO THE WAVE EQUATION IS:

$$u(x,0) = e^{i\sqrt{\ell}x}$$

$$= e^{in\frac{2\pi}{\lambda}x}$$

THIS SOLUTION IS CONSISTENT WITH OUR ABOVE PICTURE IF $n=1$.
(i.e., PERIODIC EVERY $x=\lambda$, AND ONLY λ)

EXAMPLE: $n=2$ WOULD LOOK LIKE:



WHICH COULD BE DESCRIBED BY:

$$e^{in\frac{2\pi}{\lambda'}x} \quad \text{WITH } n=1$$

DRAWING #9: ANGULAR WAVE NUMBER 2

WE SEE THAT:

$$u(x,0) = e^{in\frac{2\pi}{\lambda}x}$$

REPRESENTS THE SPATIAL PROFILE OF A RIGHT-TRAVELLING, SINUSOIDAL WAVE WHEN $n=1$:

$$u(x,0) = e^{i\frac{2\pi}{\lambda}x}$$

WE CAN SIMPLIFY THIS EXPRESSION BY DEFINING THE ANGULAR WAVE NUMBER K:

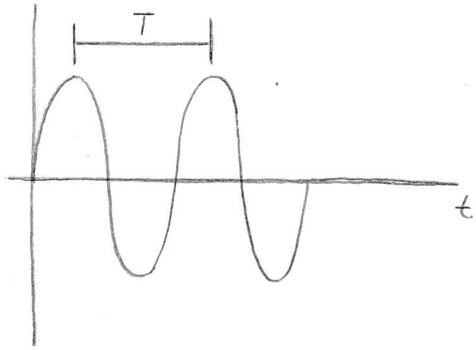
$$\boxed{k = \frac{2\pi}{\lambda}}$$

AND THEREFORE:

$$u(x,0) = e^{ikx}$$

DRAWING #10: ANGULAR FREQUENCY

CONSIDER NOW THE WAVE AS A FUNCTION OF TIME AT A FIXED x POSITION.



T: PERIOD OF OSCILLATION: TIME TAKEN FOR WAVE TO MOVE THROUGH ONE FULL OSCILLATION.

FOR SIMPLICITY (AND WITHOUT LOSS OF GENERALITY), CONSIDER $x=0$.

(ONE OF) OUR SOLUTIONS TO THE WAVE EQUATION IS:

$$u(0,t) = e^{-i\sqrt{\lambda}vt}$$

$$= e^{-i\omega_0 t}$$

WHERE $\omega_0 = \sqrt{\lambda}\, v$

$$= n\frac{2\pi}{\lambda}v$$

$$= kv \qquad (\text{FOR } n=1)$$

WHICH IS PERIODIC WITH A PERIOD OF T IF:

$$\boxed{\omega_0 = \frac{2\pi}{T}}$$

WHICH IS CALLED THE ANGULAR FREQUENCY ω_0.

DRAWING #11: WAVE SPEED

IN DERIVING THE ANGULAR FREQUENCY, WE FOUND THAT:

$$kv = \omega_0$$

THIS MEANS THAT:

$$\frac{2\pi}{\lambda} v = \frac{2\pi}{T}$$

$$\boxed{v = \frac{\lambda}{T}}$$

WHICH TELLS US THAT THE WAVE SPEED IS ONE WAVELENGTH PER PERIOD (WHICH WE MIGHT HAVE INTUITIVELY EXPECTED).

NOTE: WE'LL REVERT BACK TO THE CONSTANT $\sqrt{\ell}$, AND COME BACK TO k AND ω_0 LATER,

DRAWING #12: AMPLITUDE AND PHASE OF A WAVE · I

WE HAVE FOUND A GENERAL SOLUTION TO THE WAVE EQUATION:

$$u(x,t) = e^{i\sqrt{\ell}(x-vt)} \qquad \sqrt{\ell} = n\frac{2\pi}{\lambda}$$

WHICH DESCRIBES A RIGHT-TRAVELLING WAVE; AND FOR A FIXED n DESCRIBES A SINUSOIDAL OSCILLATION WITH WAVELENGTH λ AND PERIOD T.

HOWEVER, THIS EQUATION DESCRIBES A WAVE WITH A MAXIMUM AMPLITUDE OF 1...

... IN ADDITION, IT ASSUMES THAT THE AMPLITUDE IS 1 AT $x=0$ AND $t=0$.

WE CAN GENERALIZE OUR SOLUTION TO THE WAVE EQUATION BY MULTIPLYING IT BY A COEFFICIENT THAT CAN BE DETERMINED BY THE INITIAL CONDITIONS:

$$u(x,t) = B_n e^{i\sqrt{\ell}(x-vt)}$$

WHERE:

$$B_n = A_n e^{i\phi_n}$$

A_n: MAXIMUM AMPLITUDE OF THE WAVE

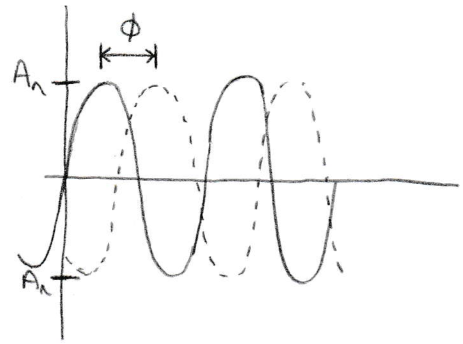
ϕ_n: PHASE CONSTANT OF THE WAVE

THEREFORE:

$$u(x,t) = A_n e^{i\phi} e^{i\sqrt{\ell}(x-vt)}$$
$$= A_n e^{i\left(\sqrt{\ell}(x-vt) + \phi_n\right)}$$

DRAWING #13: AMPLITUDE AND PHASE OF A WAVE 2

GRAPHICALLY, THE PRIOR EXPRESSION DESCRIBES A WAVE THAT LOOKS LIKE:



NOTE: THE PHASE CONSTANT HAS THE EFFECT OF SHIFTING THE WAVE OVER.

DRAWING #. 14 : COMPLETE GENERAL SOLUTION TO THE WAVE EQUATION

WE HAVE FOUND THAT THE GENERAL SOLUTION TO THE WAVE EQUATION CAN BE WRITTEN AS:

$$\boxed{u_n(x,t) = A_n\, e^{i\left(\sqrt{\ell}(x-vt) + \phi_n\right)}}$$

WHICH FOR A FIXED n DESCRIBES A RIGHT-TRAVELLING WAVE WITH A WAVELENGTH λ AND PERIOD T, WITH A MAXIMUM AMPLITUDE A_n, AND PHASE CONSTANT (SHIFT) OF ϕ_n.

SINCE THE SUPERPOSITION PRINCIPLE HOLDS, SUMS OF THE ABOVE SOLUTION ARE ALSO SOLUTIONS TO THE WAVE EQUATION.

WE CAN THEREFORE WRITE THE COMPLETE GENERAL SOLUTION TO THE WAVE EQUATION (FOR A RIGHT-TRAVELLING WAVE) AS:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n\, e^{i\left(\sqrt{\ell}(x-vt) + \phi_n\right)}} \qquad \sqrt{\ell} = n\frac{2\pi}{\lambda}$$

SINCE WE HAVE NOT MADE ANY ASSUMPTIONS ON THE SHAPE OF THE WAVE, WE SEE THAT THE ABOVE RESULT IS COMPLETELY GENERAL:

WE CAN REPRESENT ANY WAVE(-LIKE) FUNCTION AS A SUM (POSSIBLY INFINITE) OF SINE WAVES.

(NOTE: THIS IS THE ESSENCE OF THE FOURIER SERIES IN MATHEMATICS)