DRAWING #1: LC CIRCUITS!

LC CIRCUITS

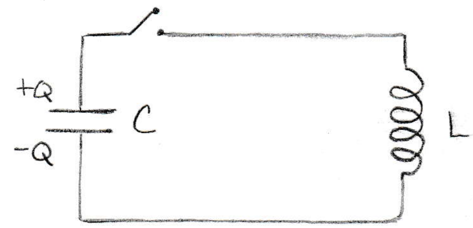
ONE TYPE OF CIRCUIT THAT EXHIBITS AN ALTERNATING CURRENT, WITHOUT REQUIRING AN ALTERNATING CURRENT SOURCE, IS THE LC CIRCUIT.

AN LC CIRCUIT CONSISTS OF AN INDUCTOR AND A CAPACITOR IN PARALLEL.

NOTE: AS WE WILL SEE, THIS SIMPLE CIRCUIT IS A KEY COMPONENT IN TELECOMMUNICATIONS --- e.g., THE OPERATION OF RADIOS, TELEVISIONS, CELL PHONES, ETC.

DRAWING #2: LC CIRCUITS 2

CONSIDER THE FOLLOWING CIRCUIT:



UP UNTIL TIME $t = t_0$, THE SWITCH HAS BEEN OPEN FOR A LONG TIME;

- THERE IS NO CURRENT FLOW
- THE VOLTAGE ACROSS THE CAPACITOR IS $V_C = Q/C$
- THERE IS NO VOLTAGE ACROSS THE INDUCTOR

AT TIME $t = t_0$, THE SWITCH IS THEN CLOSED;

- CURRENT WILL BEGIN TO FLOW
- THE VOLTAGE ACROSS THE CAPACITOR IS STILL $V_C = Q/C$ (IF WE TAKE Q TO BE ITS INSTANTANEOUS CHARGE)
- BECAUSE THE CURRENT IS CHANGING IN TIME, THERE WILL BE A VOLTAGE ACROSS THE INDUCTOR $V_L = -L \, dI/dt$

WE CAN ANALYZE THIS CIRCUIT USING THE KIRCHHOFF LOOP LAW:

$$V_{CLOSED} = \sum_i V_i = 0$$

$$V_C + V_L = 0$$

$$\frac{Q}{C} - L\frac{dI}{dt} = 0$$

THIS LAST EXPRESSION IS ONE EQUATION FOR TWO UNKNOWNS (Q AND I).

NOTING THAT THE SECOND TERM DESCRIBES THE CURRENT FLOW THROUGH THE INDUCTOR:

$$I = \frac{dq}{dt}$$ (THE AMOUNT OF CHARGE dq PASSING THROUGH THE INDUCTOR PER UNIT TIME dt)

THE RATE (OR AMOUNT) OF CHARGE THAT MOVES THROUGH THE INDUCTOR MUST BE EQUAL TO THAT WHICH LEAVES THE CAPACITOR. WE CAN THEREFORE WRITE:

$$I = \frac{dq}{dt} = -\frac{dQ}{dt}$$

WE CAN THEREFORE WRITE THE KIRCHHOFF LOOP LAW AS:

$$\frac{Q}{C} - L\frac{d}{dt}I = 0$$

$$\frac{Q}{C} - L\frac{d}{dt}\left(-\frac{d}{dt}Q\right) = 0$$

$$\frac{Q}{C} + L\frac{d^2}{dt^2}Q = 0$$

DRAWING # 4: KIRCHHOFF'S LOOP LAW (APPLIED TO LC CIRCUITS) 2

REARRANGING THE PRIOR EXPRESSION:

$$\frac{Q}{C} + L\frac{d^2}{dt^2}Q = 0$$

$$L\frac{d^2}{dt^2}Q + \frac{Q}{C} = 0$$

$$\frac{d^2}{dt}Q + \frac{1}{LC}Q = 0$$

WE CAN SIMPLIFY THE PRIOR EXPRESSION BY DEFINING THE <u>RESONANT ANGULAR FREQUENCY ω_0</u> (WE'LL SEE WHY IT IS CALLED THIS SHORTLY):

$$\boxed{\omega_0 = \left(\frac{1}{LC}\right)^{1/2}}$$

THEN:

$$\frac{d^2}{dt^2}Q + \omega_0^2 Q = 0$$

DRAWING #5: CHARGE IN LC CIRCUITS I

THE EQUATION:

$$\frac{d^2}{dt^2} Q + \omega_0^2 Q = 0$$

IS A LINEAR SECOND-ORDER DIFFERENTIAL EQUATION.

FOR THIS (SIMPLE) EQUATION, WE CAN GET A SOLUTION SIMPLY BY INSPECTION; E.G., AN EXPONENTIAL FUNCTION HAS THE PROPERTY THAT ITS DERIVATIVE IS A CONSTANT MULTIPLE OF ITSELF:

$$\frac{d}{dt} e^{st} = s e^{st} \qquad \frac{d^2}{dt^2} e^{st} = s^2 e^{st}$$

USING THE SOLUTION e^{st} IN OUR PRIOR EQUATION:

$$\frac{d^2}{dt^2}\left(e^{st}\right) + \omega_0^2\left(e^{st}\right) = 0$$

$$s^2\left(e^{st}\right) + \omega_0^2\left(e^{st}\right) = 0$$

$$s^2 + \omega_0^2 = 0$$

THE LATTER POLYNOMIAL HAS SOLUTIONS:

$$s = \pm i\omega_0 \qquad \left(i = \sqrt{-1}\right)$$

THEREFORE THE COMPLETE SOLUTION TO OUR EQUATION FOR Q IS:

$$Q = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$$

(A AND B ARE PREFACTORS TO SATISFY INITIAL CONDITIONS)

NOTE: THIS SOLUTION REPRESENTS A SINUSOIDAL ALTERNATING CHARGE (WHICH WILL BECOME MORE CLEAR SHORTLY).

OUR EXPRESSION FOR Q IS (SO FAR) COMPLEX. SINCE THE CHARGE IS A PHYSICAL QUANTITY THOUGH, IT MUST BE REAL VALUED. THIS MEANS THAT:

$$A = B^*$$ (*: COMPLEX CONJUGATE)

(NOTE: WE'LL SEE HOW THIS WORKS OUT BELOW)

WE'LL THEREFORE WRITE THEM AS GENERAL COMPLEX EXPRESSIONS, WITH AN AMPLITUDE PROPORTIONAL TO THE MAXIMUM CHARGE Q_0:

$$A = \frac{1}{2} Q_0 e^{i\phi}$$

$$B = \frac{1}{2} Q_0 e^{-i\phi}$$

NOTE: A AND B DESCRIBE THE INITIAL CHARGE ON THE CAPACITOR.

NOTE: ϕ IS THE "PHASE" --- WE'LL SEE THIS SHORTLY.

INSERTING THESE INTO OUR EXPRESSION FOR Q:

$$Q = A e^{i\omega_0 t} + B e^{-i\omega_0 t}$$

$$= \frac{1}{2} Q_0 e^{i\phi} e^{i\omega_0 t} + \frac{1}{2} Q_0 e^{-i\phi} e^{-i\omega_0 t}$$

$$= \frac{1}{2} Q_0 \left(e^{i\phi} e^{i\omega_0 t} + e^{-i\phi} e^{-i\omega_0 t} \right)$$

$$= \frac{1}{2} Q_0 \left(e^{i(\omega_0 t + \phi)} + e^{-i(\omega_0 t + \phi)} \right)$$

EULER'S FORMULA STATES THAT FOR ANY REAL NUMBER x:

$$e^{ix} = \cos x + i \sin x$$

WHICH MEANS THAT:

$$\cos x = Re\{e^{ix}\} = \frac{e^{ix} + e^{-ix}}{2}$$

NOTE: PRIOR TO THE WORK OF EULER (e.g., ABOVE) AND GAUSS (E.g., FROM GAUSS'S LAW), IMAGINARY NUMBERS WERE THOUGHT TO BE FICTITIOUS OR USELESS; ABOVE WE SAW THAT THEY WERE USEFUL TO SOLVE OUR DIFFERENTIAL EQUATION.

DRAWING #7: CHARGE IN LC CIRCUITS 3

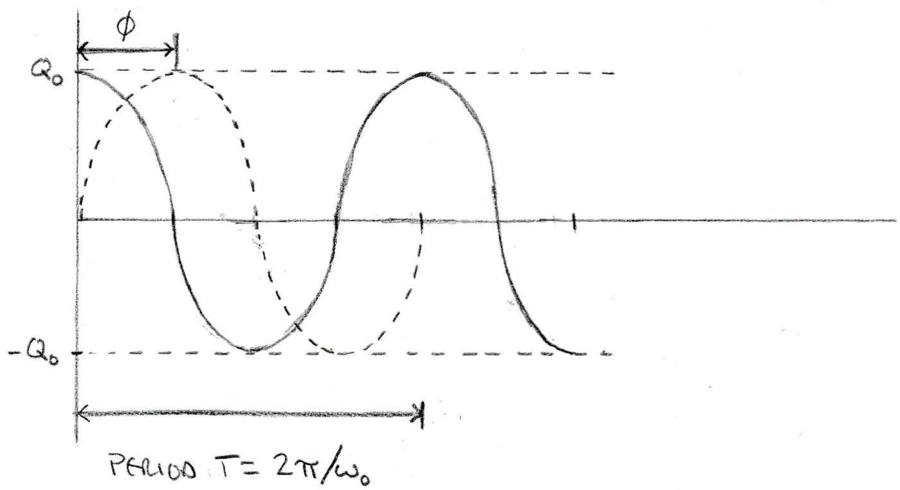
USING EULER'S FORMULA IN OUR LAST RESULT:

$$Q = Q_0 \frac{e^{i(\omega_0 t + \phi)} + e^{-i(\omega_0 t + \phi)}}{2}$$

$$\boxed{Q = Q_0 \cos(\omega_0 t + \phi)}$$

THIS SHOWS THAT THE CAPACITOR CHARGE OSCILLATES SINUSOIDALLY BETWEEN $+Q_0$ AND $-Q_0$, WITH AN ANGULAR FREQUENCY OF ω_0. THE PHASE ϕ CAN BE THOUGHT OF AS THE FRACTION OF THE WAVE THAT HAS ELAPSED RELATIVE TO THE ORIGIN.

GRAPHICALLY:



NOTE: THE SOLID LINE IS FOR A PHASE OF $\phi = 0$.

DRAWING #8: CURRENT IN LC CIRCUITS

AS THE CAPACITOR CHARGE OSCILLATES, SO DOES THE CURRENT THROUGH THE INDUCTOR:

$$I = -\frac{dQ}{dt}$$

$$= -\frac{d}{dt}\left(Q_0 \cos(\omega_0 t + \phi)\right)$$

$$= \omega_0 Q_0 \sin(\omega_0 t + \phi)$$

FROM WHICH WE CAN IDENTIFY THE MAXIMUM CURRENT:

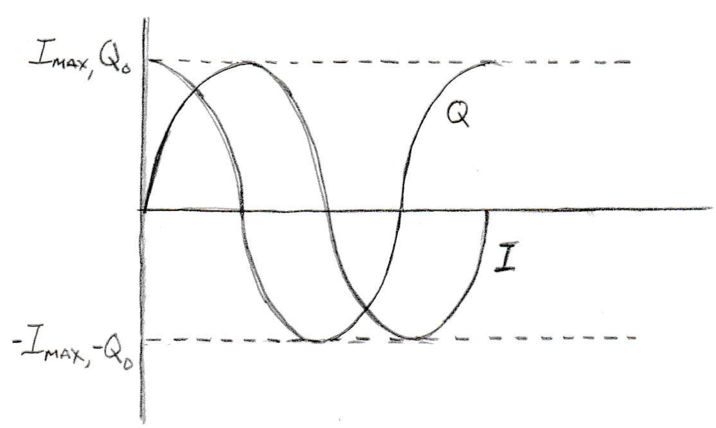
$$\boxed{I_{MAX} = \omega_0 Q_0}$$

... AND WRITE:

$$\boxed{I = I_{MAX} \sin(\omega_0 t + \phi)}$$

NOTE: COMPARING THIS EXPRESSION TO THAT FOR Q SHOWS THAT THEY ARE OUT OF PHASE (THE MAGNITUDE OF I IS AT A MAXIMUM WHEN THAT OF Q IS AT A MINIMUM, AND VICE VERSA).

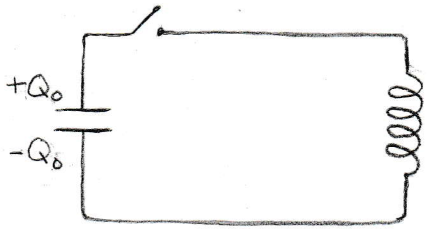
GRAPHICALLY:



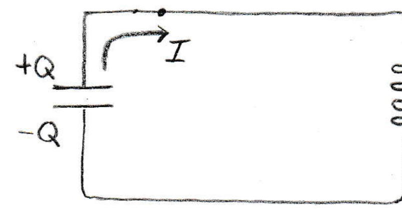
NOTE: I AND Q ARE SHOWN FOR $\phi = 0$.

DRAWING #9: DYNAMICS OF AN LC CIRCUIT

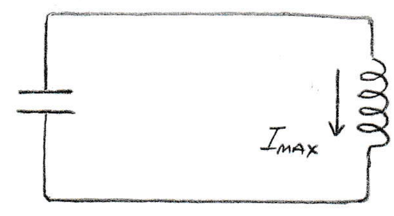
THE DYNAMICS OF AN LC CIRCUIT LOOK LIKE:



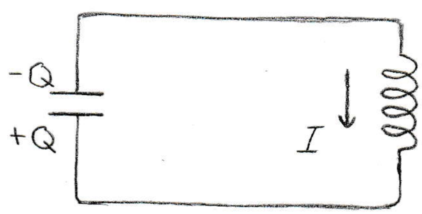
UP TO TIME $t = t_0$: THE CAPACITOR PLATES ARE CHARGED TO $+Q_0 / -Q_0$; AND THE SWITCH IS OPEN, SO NO CURRENT FLOWS.



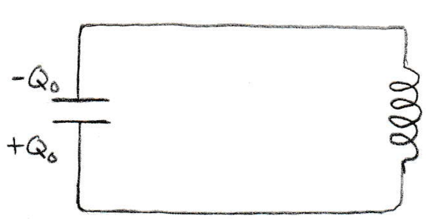
AT TIME $t = t_0$: THE SWITCH IS CLOSED; THE CURRENT BEGINS TO FLOW (INCREASES WITH TIME), AS CHARGE IS TRANSFERED BETWEEN THE CAPACITOR PLATES.



AT TIME $t = \pi / 2\omega_0$: THE CURRENT THROUGH THE INDUCTOR IS AT A MAXIMUM (I_{MAX}); AT THE SAME TIME, THE CHARGE ON THE CAPACITOR PLATES IS ZERO.



AT TIME $t' > t$: CURRENT CONTINUES TO FLOW, BUT DECREASES WITH TIME; THE EFFECT OF THIS IS THAT CHARGE BUILDS UP ON THE CAPACITOR PLATES, OPPOSITE TO THE STARTING CONFIGURATION.



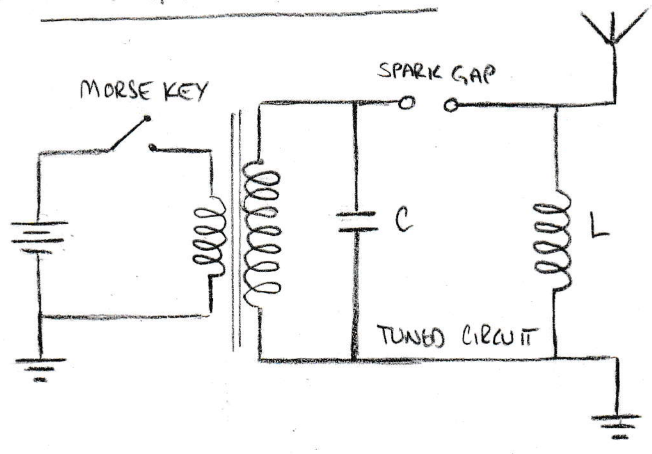
AT TIME $t = \pi / \omega_0$: THE CURRENT FLOW CEASES; CHARGE HAS BEEN COMPLETELY REVERSED

THE PROCESS THEN REPEATS (IN REVERSE), AND REPEATS, ETC. THE CURRENT OSCILLATES AT THE CIRCUIT'S NATURAL OSCILLATION FREQUENCY ω_0.

DRAWING #10: EXAMPLE: SPARK-GAP TRANSMITTER

AS MENTIONED EARLIER, LC CIRCUITS ARE KEY COMPONENTS IN TELECOMMUNICATION DEVICES.

EXAMPLE: THE FIRST DEVICE TO DEMONSTRATE PRACTICAL RADIO TRANSMISSION WAS A SPARK-GAP TRANSMITTER.



TOP PORTION OF AN AERIAL (ANTENNA): USED TO TRANSMIT THE SIGNAL

GROUND WIRE: KEEPS THE REFERENCE POTENTIAL OF THE TWO CIRCUITS THE SAME AND CONSTANT

WHEN THE MORSE KEY (SIMILAR TO A SWITCH) IS PRESSED, CURRENT WILL FLOW THROUGH THE LEFT CIRCUIT...

... THIS WILL CREATE A MAGNETIC FIELD THROUGH THE INDUCTOR (LEFT CIRCUIT)...

... RECALLING THAT MAGNETIC FIELDS FORM CLOSED LOOPS, THE RESULTING FIELD WILL PASS THROUGH THE INDUCTOR COIL IN THE RIGHT CIRCUIT...

... THIS WILL CAUSE CHARGE TO BUILD UP ON THE CAPACITOR (AND HENCE THERE WILL BE A VOLTAGE ACROSS IT)...

... WHEN THE VOLTAGE IS GREAT ENOUGH, A SPARK WILL BRIDGE THE GAP (IN THE SPARK GAP) MEANING THAT THE GAS BETWEEN THE ENDS WILL BECOME IONIZED, REDUCING ITS RESISTANCE, AND CURRENT WILL FLOW IN THE "TUNED CIRCUIT" (AN LC CIRCUIT, WHICH WILL OSCILLATE AT ITS NATURAL FREQUENCY)...

... THE CURRENT IN THIS CIRCUIT WILL OSCILLATE AT THE CIRCUIT'S NATURAL FREQUENCY...

... AND THIS CURRENT WILL EXTEND TO THE AERIAL, WHICH WILL EMIT ELECTROMAGNETIC RADIATION (RECALL MAXWELL'S EQUATIONS), AT THE FREQUENCY OF THE LC CIRCUIT.