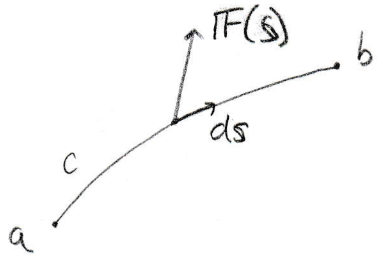
DRAWING #1: ELECTRIC POTENTIAL ENERGY (RECAP)

RECALL THAT WE ARE CONSIDERING THE MOTION OF A POINT CHARGE THAT MOVES IN THE PRESENCE OF OTHER CHARGE(S) OR A CHARGE DISTRIBUTION.



$\mathbb{F}(s)$: THE TOTAL ELECTRICAL FORCE ON THE POINT CHARGE.

IT HAS BEEN SHOWN THAT SINCE THE WORK W DONE BY/AGAINST THE ELECTRICAL FORCE IS INDEPENDENT OF THE PATH...

$$W = \int_a^b \mathbb{F}(s) \cdot ds$$

ANY PATH

... IT CAN BE REPRESENTED AS THE DIFFERENCE IN A SCALAR FUNCTION EVALUATED AT THE END POINTS:

$$\int_a^b \mathbb{F}(s) \cdot ds = -\left(\mathcal{U}(b) - \mathcal{U}(a)\right)$$

ANY PATH

WHERE $\mathcal{U}$ IS THE ELECTRIC POTENTIAL ENERGY (OF INTERACTION).

AN IMPORTANT RESULT OF THE PRIOR EQUATION IS THAT ONCE A REFERENCE POINT (a) HAS BEEN CHOSEN, THE ELECTRIC POTENTIAL ENERGY OF A CHARGE WITH OTHER CHARGE(S) AND/OR A CHARGE DISTRIBUTION CAN BE UNIQUELY CALCULATED BY PLACING THAT CHARGE AT ANY POINT IN SPACE.

... HOWEVER, THIS IS STILL A LOCAL EFFECT; $\mathcal{U}$ IS DEFINED FOR A CHARGE PLACED AT A GIVEN POINT.

DRAWING #2: WORK PER UNIT CHARGE

THE PRIOR RESULTS SUGGEST A MATHEMATICAL WAY TO DESCRIBE THE REGION AROUND CHARGE(S) AND/OR A CHARGE DISTRIBUTION, INDEPENDENT OF THE (TEST) POINT CHARGE...

... WE CAN EXTEND OUR PRIOR RESULTS TO THOSE "PER UNIT CHARGE"

NOTE: THIS IS THE SAME APPROACH THAT WE TOOK TO DEFINE THE ELECTRIC FIELD (THE ELECTRICAL FORCE PER UNIT CHARGE --- DEFINED AT EVERY POINT IN SPACE)

THE <u>WORK DONE BY THE ELECTRICAL FORCE PER UNIT CHARGE</u>:

$$W = \int_{a}^{b} \mathbb{F}(s) \cdot ds$$

ANY PATH

HENCE:

$$W(\text{UNIT}) = \int_{a}^{b} \left(\mathbb{F}(s) / q \right) \cdot ds \qquad (q > 0)$$

ANY PATH

$$= \int_{a}^{b} \mathbb{E}(s) \cdot ds$$

ANY PATH

WHERE WE HAVE USED OUR PRIOR DEFINITION OF THE ELECTRIC FIELD:

$$\mathbb{E}(x,y,z) = \frac{\mathbb{F}(x,y,z)}{q}$$

RECALL THAT $\mathbb{E}$ IS A <u>VECTOR FIELD</u> (THE ASSIGNMENT OF A VECTOR TO EACH POINT IN SPACE), AND IS INDEPENDENT OF q; HENCE W (UNIT) IS A SCALAR INDEPENDENT OF q.

DRAWING #3: ELECTRIC POTENTIAL

SINCE THE WORK PER UNIT CHARGE IS INDEPENDENT OF PATH (SINCE THE WORK ON A UNIT CHARGE IS), IT TOO CAN BE EXPRESSED AS A DIFFERENCE IN A SCALAR FUNCTION EVALUATED AT THE ENDPOINTS:

$$W\,(\text{unit}) = \underset{\text{ANY PATH}}{\int_a^b} \mathbb{E}(s)\cdot ds = -\left(\phi(b)-\phi(a)\right)$$

WHERE ϕ IS THE (SCALAR) ELECTRIC POTENTIAL (NOT TO BE CONFUSED WITH THE ELECTRIC POTENTIAL ENERGY $\mathcal{U}$). NOTE THAT SIMILAR TO $\mathcal{U}$, ϕ IS DEFINED AS THE WORK PER UNIT CHARGE AGAINST A FORCE, SO THAT POSITIVE WORK LEADS TO A REDUCTION IN THE POTENTIAL.

SINCE ONLY DIFFERENCES IN ϕ BETWEEN ENDPOINTS ARE INVOLVED, ONCE A REFERENCE POINT $(\phi(a))$ HAS BEEN CHOSEN (OFTEN AT INFINITY, WITH $\lim_{a\to\infty}\phi(a)=0$), ϕ IS (UNIQUELY) DETERMINED AT EVERY POINT IN SPACE $(\phi(b))$ --- IT IS A SCALAR FIELD (THE ASSIGNMENT OF A SCALAR TO EVERY POINT (x,y,z)).

DRAWING #4: THE ELECTRIC POTENTIAL AND ELECTRIC POTENTIAL ENERGY

IT IS CLEAR THAT ϕ AND U ARE RELATED BY:

$$\phi(x,y,z) = \frac{U(x,y,z)}{q_{(x,y,z)}}$$

AN IMPORTANT DISTINCTION, HOWEVER, IS THAT $U(x,y,z)$ IS DEFINED IN TERMS OF THE CHARGE $q_{(x,y,z)}$ AT (x,y,z) (i.e., IT IS LOCAL); $\phi(x,y,z)$ IS A FUNCTION DEFINED AT ANY POINT IN SPACE.

DRAWING #8.1: VOLTS

RECALL THAT ELECTRIC POTENTIAL IS THE ELECTRIC POTENTIAL ENERGY PER UNIT CHARGE.

$$\phi(1) = \frac{U(1)}{q}$$

THE SI (DERIVED) UNIT OF ELECTRIC POTENTIAL IS A <u>VOLT (V)</u>:

$$V = \frac{\text{POTENTIAL ENERGY}}{\text{CHARGE}}$$

... AND DEFINED SO THAT 1 V IS EQUAL TO THE POTENTIAL DIFFERENCE BETWEEN TWO PARALLEL INFINITE PLANES SPACED 1 m APART THAT CREATE AN ELECTRIC FIELD OF 1 N/C:

$$V = \frac{N \cdot m}{C}$$

(NOTE: IN TERMS OF VOLTS, AN ELECTRIC FIELD HAS UNITS OF V/m)

IN TERMS OF ELECTRICAL CIRCUITS, A VOLT CAN ALSO BE DEFINED AS THE DIFFERENCE IN ELECTRIC POTENTIAL BETWEEN TWO POINTS OF A CONDUCTING WIRE WHEN AN ELECTRIC CURRENT DISSIPATES ONE WATT OF POWER BETWEEN THE POINTS.

(POWER IS THE RATE OF DOING WORK --- IN THIS CASE, IN MOVING CHARGES BETWEEN TWO POINTS)

$$V = \frac{W}{A}$$

A DIFFERENCE IN ELECTRIC POTENTIAL BETWEEN TWO POINTS IS CALLED A <u>VOLTAGE V</u>:

$$V = \phi(a) - \phi(b)$$

DRAWING #5: THE ELECTRIC POTENTIAL IN VARIOUS CIRCUMSTANCES

WE CAN EASILY EXTEND OUR PRIOR RESULTS FOR U IN VARIOUS CIRCUMSTANCES TO ϕ:

THE ELECTRIC POTENTIAL AT A DISTANCE r FROM A POINT CHARGE:

$$\phi(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

THE ELECTRIC POTENTIAL FROM MULTIPLE POINT CHARGES:

$$\phi(1) = \sum_i \frac{1}{4\pi\varepsilon_0}\frac{q_i}{r_{1i}}$$

r_{1i}: THE DISTANCE FROM POINT CHARGE i TO (1).

THE ELECTRIC POTENTIAL FROM A CHARGE DISTRIBUTION:

$$\phi(1) = \iiint_D dx\, dy\, dz \frac{1}{4\pi\varepsilon_0}\frac{\rho(x,y,z)}{r_{1(x,y,z)}}$$

NOTE: IN ALL CASES, A REFERENCE POINT AT INFINITY WAS ASSUMED.

DRAWING #6: GRADIENT THEOREM

RECALL THE SECOND FUNDAMENTAL THEOREM OF CALCULUS:

LET f AND F BE REAL-VALUED FUNCTIONS SUCH THAT FOR ALL x IN $[a, b]$:

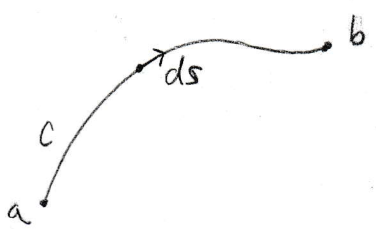
$$F'(x) = f(x)$$

IF f IS RIEMANN INTEGRABLE ON $[a, b]$:

$$\int_a^b dx\, f(x) = F(b) - F(a)$$

(THAT IS, THE THEOREM PROVIDES A LINK BETWEEN THE CONCEPT OF THE DERIVATIVE OF A FUNCTION WITH THE CONCEPT OF THAT FUNCTION'S INTEGRAL.)

THE GRADIENT THEOREM GENERALIZES THE FUNDAMENTAL THEOREM OF CALCULUS FROM THE REAL LINE (x) TO ANY CURVE:



$$\int_C \nabla \varphi(s) \cdot ds = \varphi(b) - \varphi(a)$$

$$\nabla f = \frac{\partial f}{\partial e_1} e_1 + \ldots + \frac{\partial f}{\partial e_n} e_n$$

(THE GRADIENT: THE GENERALIZATION OF THE DERIVATIVE OF A FUNCTION TO SEVERAL DIMENSIONS)

TWO IMPORTANT MATHEMATICAL CONSEQUENCES:

- LINE INTEGRALS THROUGH GRADIENT FIELDS ARE PATH INDEPENDENT
- ANY PATH-INDEPENDENT VECTOR-FIELD CAN BE REPRESENTED AS THE GRADIENT OF A SCALAR FIELD

DRAWING #7: THE CONNECTION BETWEEN ELECTRIC FIELD AND ELECTRIC POTENTIAL

RECALL THE WORK PER UNIT CHARGE:

$$W(\text{UNIT}) = \int_{a \atop \text{ANY PATH}}^{b} \mathbb{E}(s) \cdot ds = -\left(\phi(b) - \phi(a)\right)$$

FROM THE GRADIENT THEOREM:

$$\phi(b) - \phi(a) = \int_{a \atop \text{ANY PATH}}^{b} \nabla\phi(s) \cdot ds$$

SO:

$$\int_{a \atop \text{ANY PATH}}^{b} \mathbb{E}(s) \cdot ds = -\int_{a \atop \text{ANY PATH}}^{b} \nabla\phi(s) \cdot ds$$

$$\int_{a \atop \text{ANY PATH}}^{b} \mathbb{E}(s) \cdot ds + \int_{a \atop \text{ANY PATH}}^{b} \nabla\phi(s) \cdot ds = 0$$

$$\int_{a \atop \text{ANY PATH}}^{b} \left(\mathbb{E}(s) + \nabla\phi(s)\right) \cdot ds = 0$$

$$\mathbb{E}(s) + \nabla\phi(s) = 0$$

HENCE:

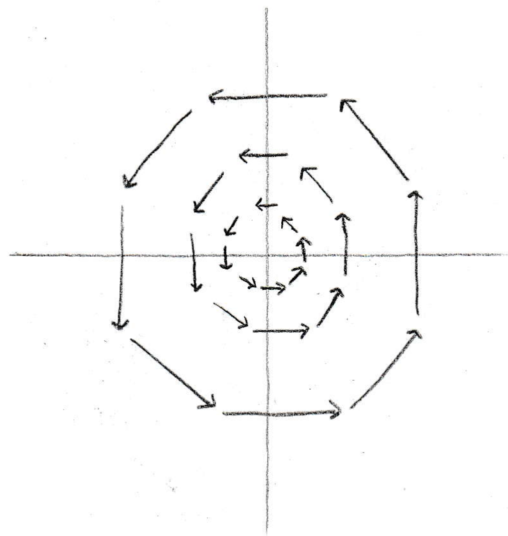
$$\boxed{\mathbb{E}(s) = -\nabla\phi(s)}$$

THE ELECTRIC FIELD (A VECTOR) IS THEREFORE EQUAL TO THE GRADIENT OF THE ELECTRIC POTENTIAL (A SCALAR) --- IN OTHER WORDS, IT ONLY TAKES A SINGLE SCALAR FIELD ϕ TO DEFINE THE VECTOR FIELD $\mathbb{E}$.

DRAWING #8: $\mathbb{E}$ IS A CURL-FREE FIELD!

CONSIDER THE <u>CURL</u> OF A VECTOR FIELD; THE CURL OPERATOR $(\nabla \times)$ DESCRIBES THE INFITESIMAL ROTATION OF A VECTOR FIELD.

EXAMPLE: $\mathbb{F}(x,y,z) = -y\,e_x + x\,e_y$



$\nabla \times \mathbb{F} = 2\,e_z$

$\uparrow$

THAT IS, $\mathbb{F}$ HAS CURL WITH POSITIVE e_z COMPONENT.

FROM VECTOR CALCULUS:

$$\nabla \times (\nabla \varphi) = 0 \qquad \text{FOR ANY } \varphi$$

(THE AMOUNT OF ROTATION ALONG THE GRADIENT OF A SCALAR FIELD IS ZERO)

THEREFORE:

$$\mathbb{E}(x,y,z) = -\nabla \Phi(x,y,z)$$

$$\nabla \times \mathbb{E}(x,y,z) = -\nabla \times \left(\nabla \Phi(x,y,z)\right) = 0$$

$$\boxed{\nabla \times \mathbb{E}(x,y,z) = 0}$$

THE ELECTRIC FIELD (IN ELECTROSTATICS) IS A <u>CURL-FREE FIELD</u>. THIS IS OUR <u>SECOND FUNDAMENTAL EQUATION OF ELECTROSTATICS</u>.

DRAWING #9: $\mathbb{E}$ IS A CURL-FREE FIELD 2

NOTE THAT:

$$\nabla \times \mathbb{E}(\mathbb{S}) = 0$$

IS ANOTHER WAY OF SAYING:

$$W(\text{UNIT}) = \oint \mathbb{E}(\mathbb{S}) \cdot d\mathbb{s} = 0$$

NOTE THAT, AS WITH GAUSS'S LAW, THIS EQUATION <u>ALONE</u> IS NOT ENOUGH TO SOLVE ELECTROSTATIC PROBLEMS.