DRAWING #1: ELECTROSTATICS

COULOMB'S LAW PLUS THE SUPERPOSITION PRINCIPLE ARE ALL THERE IS TO ELECTROSTATICS ...

... WITH OUR EQUATIONS FOR $\mathbb{F}$ AND $\mathbb{E}$, WE HAVE SOLVED ALL ELECTROSTATIC PROBLEMS FOR WHICH WE KNOW THE LOCATIONS OF ALL OF THE CHARGES OR THE CHARGE DISTRIBUTION(S):

$$\mathbb{F}_1 = \frac{1}{4\pi\varepsilon_0} q_1 \iiint_D dx\,dy\,dz \; \frac{\rho(x,y,z)}{r^2_{1(x,y,z)}} \; e_{1(x,y,z)}$$

$$\mathbb{E}(1) = \frac{\mathbb{F}_1}{q_1}$$

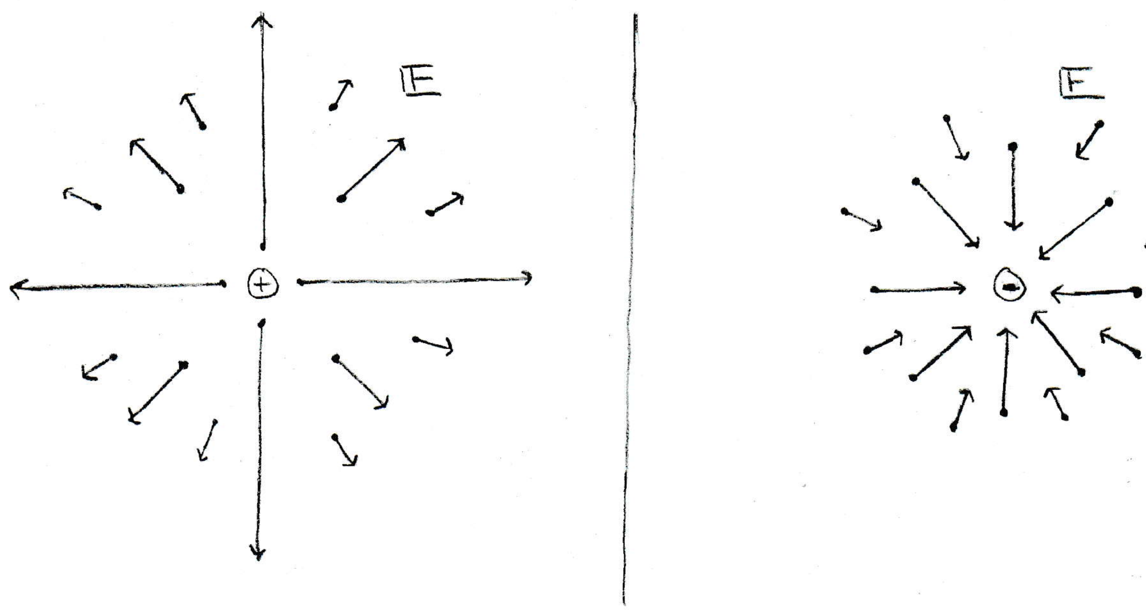
RATHER THAN COMPUTE THESE INTEGRALS DIRECTLY, IT IS SOMETIMES EASIER TO COMPUTE THEM - BY SOME CLEVER GUESSWORK.

IN DOING SO, WE WILL ALSO BE ABLE TO STATE COULOMB'S LAW AND THE EQUATION FOR THE ELECTRIC FIELD IN MORE FUNDAMENTAL (GENERAL, AND ABSTRACT) WAYS.

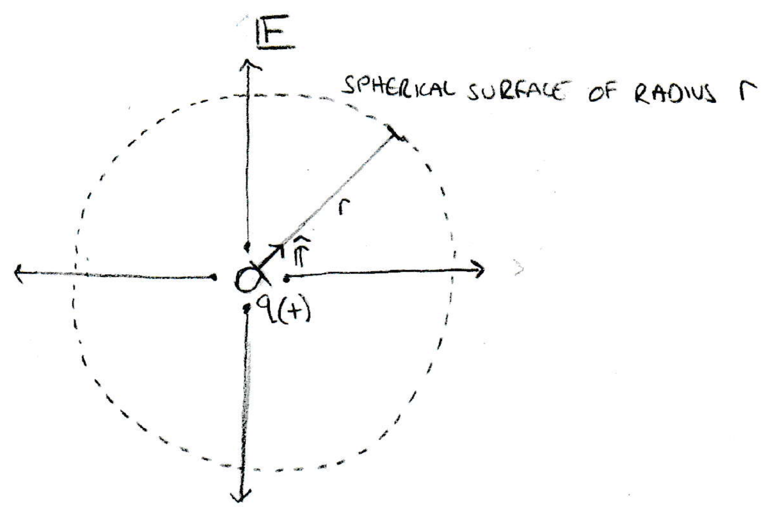
KEEP IN MIND THOUGH AS WE DO SO, WE ALREADY HAVE THE ANSWER HERE!

DRAWING #3: THE ELECTRIC FIELD DUE TO A POINT CHARGE 2
(FROM LECTURE 4)

THE ELECTRIC FIELD OF POSITIVE AND NEGATIVE POINT CHARGES:



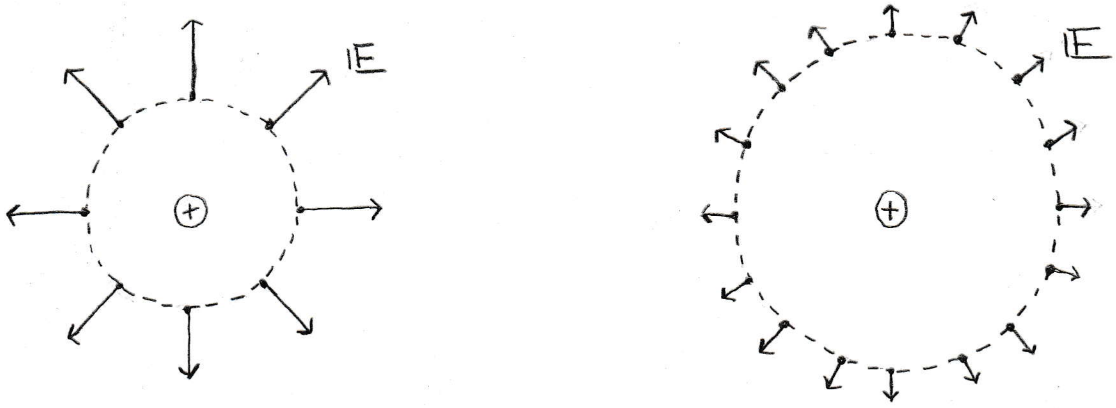
DUE TO THE SPHERICAL SYMMETRY:



$$\mathbb{E}(\mathbb{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbb{r}}$$

DRAWING #3: THE FLOW OF THE ELECTRIC FIELD

CONSIDER WHAT $\mathbb{E}$ LOOKS LIKE, DRAWN OVER SURFACES OF DIFFERENT r:

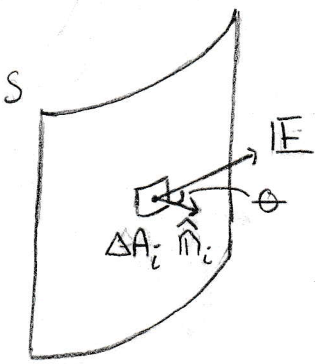


THE ELECTRIC FIELD OVER THE SURFACE ON THE LEFT IS STRONGER, BUT THE SURFACE AREA IS ALSO SMALLER...

...THIS LEADS US TO SUPPOSE THE FOLLOWING <u>MODEL</u>: THAT THE ELECTRIC FIELD REPRESENTS THE <u>FLOW</u> OF SOMETHING THAT IS CONSERVED --- i.e., IT STARTS AT A CHARGE AND FLOWS OUT INTO SPACE.

NOTE: ONE CANNOT ACTUALLY SAY THAT THE ELECTRIC FIELD MEASURES THE FLOW OF SOMETHING THAT MIGHT BE CONSERVED --- WE ARE SIMPLY USING A MODEL TO HELP US FIND THE RIGHT MATHEMATICS.

DRAWING #4: ELECTRIC FLUX



$\hat{n}_i$: THE <u>NORMAL VECTOR</u> (A UNIT VECTOR PERPENDICULAR TO THE SURFACE AT A GIVEN POINT)

WE DEFINE THE FLOW OF THE ELECTRIC FIELD THROUGH A SMALL ELEMENT i ON A SURFACE S (A <u>SURFACE ELEMENT</u>) TO BE (APPROXIMATELY) THE AREA ΔA_i TIMES THE AVERAGE VALUE OF $\mathbb{E}$ (OVER i) PERPENDICULAR TO ΔA_i:

$$\Delta \Phi_i \approx \mathbb{E} \cdot \hat{n} \, \Delta A_i \qquad \left(\text{UNITS: } N \cdot m^2/C^2\right)$$

$$\approx E_n \Delta A_i \qquad E_n = \mathbb{E} \cdot \hat{n} = (\|\mathbb{E}\|)(\|\hat{n}\|)\cos\theta$$

$\Delta \Phi_i$ IS CALLED THE <u>ELECTRIC FLUX</u> (THROUGH THE SURFACE ELEMENT i).

TO FIND THE <u>TOTAL FLUX</u> THROUGH THE SURFACE S:

$$\Phi \approx \sum_{i=1}^{N} E_n \Delta A_i$$

WE CAN MAKE THIS EXPRESSION MORE EXACT BY TAKING MAX $\Delta A_i \to 0$:

$$\Phi = \lim_{\text{MAX } \Delta A_i \to 0} \sum_{i=1}^{N} E_n \Delta A_i$$

$$= \int_S dA \, E_n$$

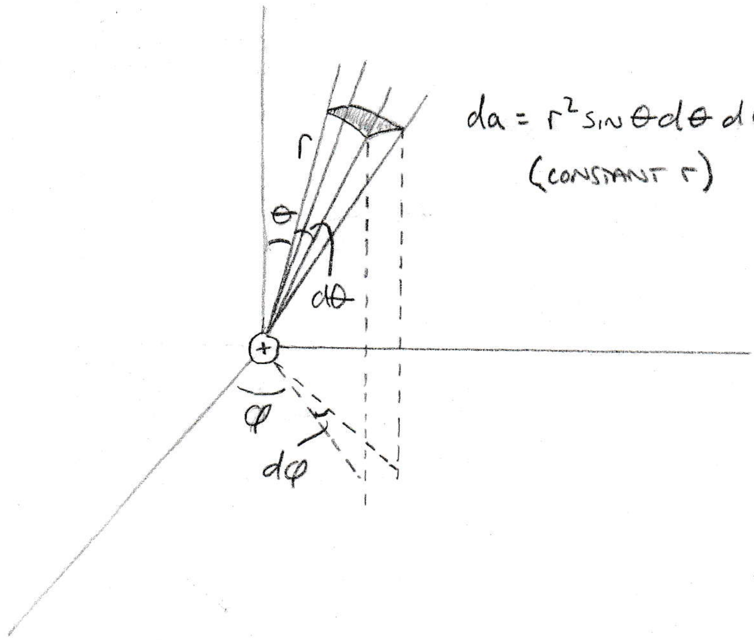
IN MANY CASES, WE'LL BE INTERESTED IN <u>CLOSED SURFACES</u>:

$$\Phi = \oint_S dA \, E_n$$

NOTE: WE MAKE THE CONVENTION THAT $\hat{n}$ (HENCE, E_n) POINTS OUTWARD.

EXAMPLE : ELECTRIC FLUX AROUND A POINT CHARGE !

QUESTION : WHAT IS THE ELECTRIC FLUX THROUGH AN AREA ELEMENT SURROUNDING A POINT CHARGE, AS SHOWN IN THE FOLLOWING FIGURE?



(STEP 1)

THE ELECTRIC FLUX THROUGH A SURFACE IS GIVEN BY:

$$\Phi = \int_S da \, E_n \qquad E_n = \mathbb{E} \cdot \hat{n}$$

WE ALSO KNOW THE ELECTRIC FIELD DUE TO A POINT CHARGE:

$$\mathbb{E}(r) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

(STEP 2)

SINCE $\mathbb{E}$ IS RADIAL, IT IS EVERYWHERE NORMAL TO OUR SURFACE ELEMENT:

$$E_n = \mathbb{E} \cdot \hat{n} \qquad (\hat{n} = \hat{r}, \ \hat{n} \cdot \hat{n} = 1)$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2}$$

EXAMPLE : ELECTRIC FLUX AROUND A POINT CHARGE 2

ADDITIONALLY, SINCE WE ARE ONLY INTERESTED IN ONE AREA ELEMENT:

$$\Phi = \int_S da\, E_n$$

$$= da\, E_n$$

(STEP 3)

PUTTING ALL OF THE PIECES TOGETHER:

$$\Phi = da\, E_n$$

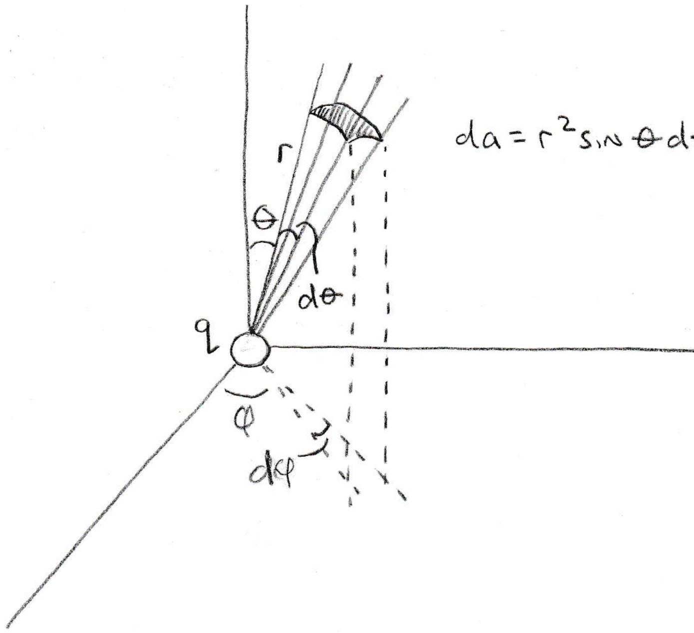
$$= \left(r^2 \sin\theta\, d\theta\, d\varphi\right)\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)$$

$$= \left(\sin\theta\, d\theta\, d\varphi\right)\left(\frac{1}{4\pi\varepsilon_0}\, q\right)$$

THE FLUX DOES NOT DEPEND ON r!

THIS INDEPENDENCE ARISES FROM THE FACT COULOMB'S LAW IS AN INVERSE-SQUARE LAW. ANY OTHER EXPONENT n IN $1/r^n$ WOULD NOT GIVE THIS.

DRAWING #1: THE ELECTRIC FLUX AROUND A POINT CHARGE!

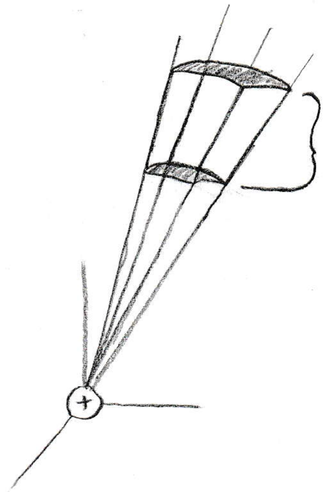


$da = r^2 \sin\theta \, d\theta \, d\varphi$

THAT THE FLUX THROUGH AN AREA ELEMENT (SPHERICAL FACE) AROUND A POINT CHARGE IS ZERO HAS IMPORTANT IMPLICATIONS:

THIS MEANS THAT THE NET FLUX THROUGH A CLOSED SURFACE (SPHERICAL FACES) SURROUNDING A POINT CHARGE IS ZERO:

$$\Phi = \oint_S da \, E_n = 0$$



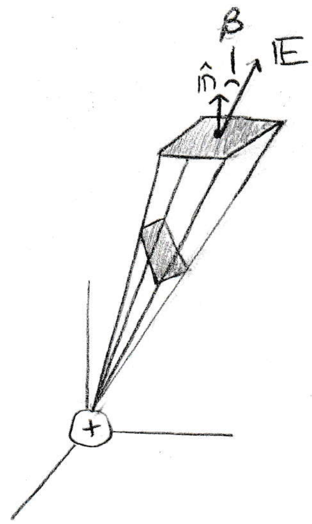
THE MAGNITUDE OF THE FLUX ON BOTH SURFACES IS THE SAME (DOES NOT DEPEND ON r); ON THE RADIAL FACES, THE NORMAL COMPONENT OF $\mathbb{E}$ IS ZERO.

THIS RESULT IS CONSISTENT WITH OUR MODEL THAT THE ELECTRIC FIELD REPRESENTS THE FLOW OF SOMETHING THAT IS CONSERVED

DRAWING #2: THE ELECTRIC FLUX AROUND A POINT CHARGE 2

IMAGINE TILTING THE SURFACES (SHOWN FOR A POSITIVE CHARGE):

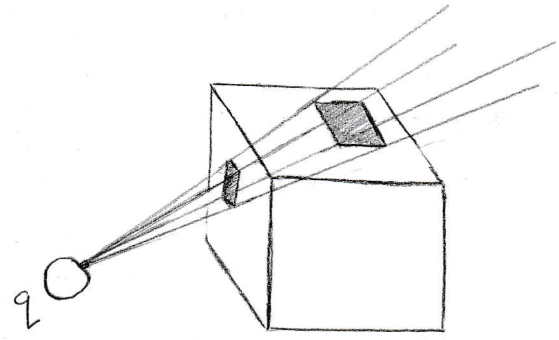
NOTE: WE ARE CONSIDERING SMALL SURFACES, SO THAT $\mathbb{E}$ CAN BE CONSIDERED UNIFORM OVER THE SURFACE.



IF WE TILT A SURFACE BY A ANGLE β, ITS AREA INCREASES BY A FACTOR $(1/\cos\beta)$; E_n, HOWEVER, DECREASES BY THE FACTOR $(\cos\beta)$. THEREFORE, $(E_n \cdot da)$ IS UNCHANGED --- i.e., THE TOTAL FLUX IS STILL ZERO.

ANY VOLUME IN THE VICINITY OF A POINT CHARGE CAN BE CONSIDERED TO BE MADE UP OF TILTED SURFACE ELEMENTS...

<u>EXAMPLE:</u>



(NOTE: SHOWN IS ONLY ONE TILTED SURFACE ELEMENT)

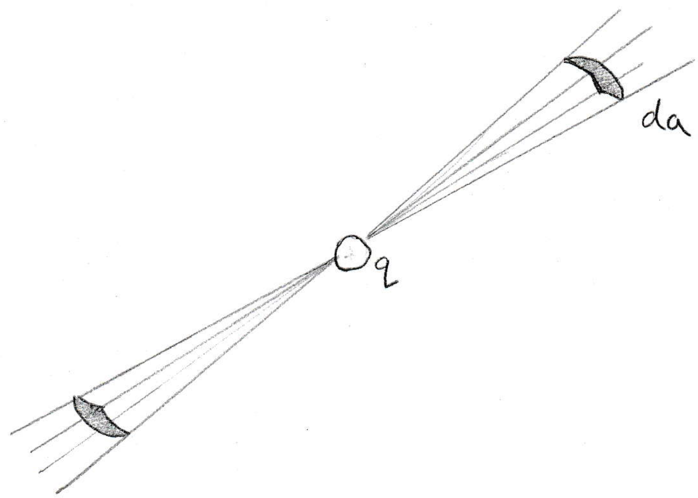
... SINCE THE TOTAL FLUX OUT OF EACH ELEMENT IS ZERO, WE HAVE THE GENERAL RESULT:

<u>THE TOTAL FLUX THROUGH ANY VOLUME IN THE VICINITY OF A POINT CHARGE IS ZERO.</u>

(NOTE: THIS ASSUMES NO OTHER POINT CHARGES ARE PRESENT)

EXAMPLE: THE ELECTRIC FLUX THROUGH A SURFACE CONTAINING A POINT CHARGE

QUESTION: WHAT IS THE ELECTRIC FLUX THROUGH A SURFACE CONTAINING A POINT CHARGE, AS SHOWN IN THE FOLLOWING FIGURE?



NOTE THAT WE HAVE ALREADY DERIVED THE FLUX THROUGH AN AREA ELEMENT da:

$$\Phi = (\sin\theta \, d\theta \, d\phi)\left(\frac{1}{4\pi\varepsilon_0} q\right)$$

AND YOU CAN CONSIDER $(\sin\theta \, d\theta \, d\phi)$ THE SAME FOR BOTH SURFACES.

(STEP 1)

NOTICE THAT, AS BEFORE, THE MAGNITUDES OF THE FLUXES THROUGH THE TWO SURFACES ARE EQUAL. HOWEVER, THEY NOW HAVE THE SAME SIGN.

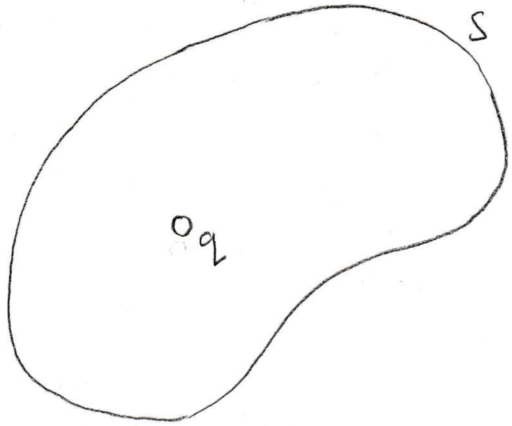
HENCE:

$$\Phi = \Phi_1 + \Phi_2$$
$$= (\sin\theta \, d\theta \, d\phi)\left(\frac{1}{2\pi\varepsilon_0} q\right)$$

THE ELECTRIC FLUX OUT OF A VOLUME CONTAINING A CHARGE IS NOT ZERO.

DRAWING #4: THE ELECTRIC FLUX THROUGH A SURFACE CONTAINING A POINT CHARGE 2

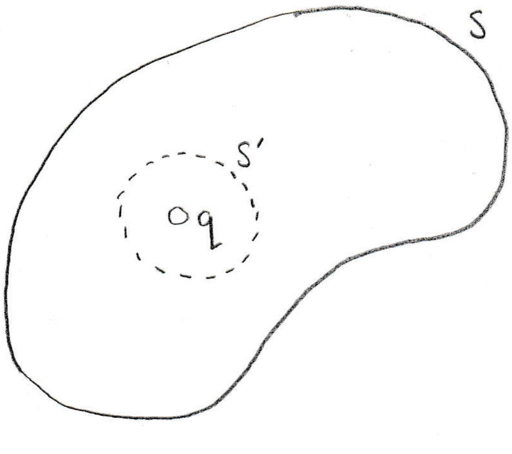
SUPPOSE THAT WE WANT TO CALCULATE THE FLUX OUT OF SOME ARBITRARY SURFACE S CONTAINING A POINT CHARGE:



NOTE: A CLOSED SURFACE THROUGH WHICH THE FLUX OF A VECTOR FIELD IS CALCULATED IS CALLED A <u>GAUSSIAN SURFACE</u>.

WE CAN SOLVE THIS PROBLEM AS FOLLOWS:

SUPPOSE THAT WE SURROUND THE CHARGE WITH A NEW SURFACE S':



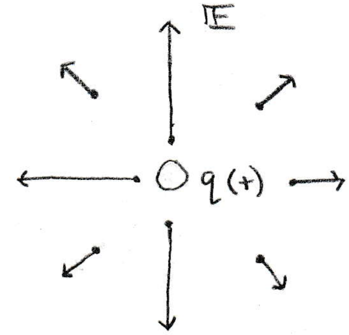
THE VOLUME BETWEEN S' AND S NOW HAS NO CHARGE IN IT. THEREFORE THE TOTAL FLUX THROUGH <u>THIS</u> VOLUME IS ZERO; THIS MEANS THAT THE TOTAL FLUX OUT OF S IS THE SAME AS THAT INTO THE VOLUME BETWEEN S' AND S VIA S' (i.e., OUT OF S').

<u>THE FLUX OUT OF A VOLUME CONTAINING A POINT CHARGE IS INDEPENDENT OF ITS SHAPE.</u>

NOTE (AGAIN): THIS IS CONSISTENT WITH OUR MODEL FOR THE ELECTRIC FIELD REPRESENTING THE FLOW OF SOMETHING THAT IS CONSERVED.

EXAMPLE: THE ELECTRIC FLUX THROUGH A SURFACE CONTAINING A POINT CHARGE

QUESTION: WHAT IS THE FLUX THROUGH A SURFACE CONTAINING A POINT CHARGE?

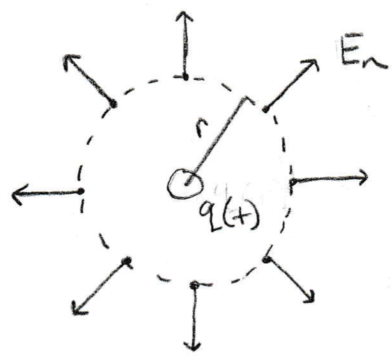


NOTE: THE ELECTRIC FIELD IS DRAWN, AS AN EXAMPE, FOR A POSITIVE POINT CHARGE.

(STEP 1)

WE HAVE SHOWN THAT WE CAN USE A CLOSED SURFACE OF ANY SHAPE TO CALCULATE THE ELECTRIC FLUX.

BECAUSE THE ELECTRIC FIELD FROM A POINT CHARGE IS SPHERICALLY SYMMETRIC, IT IS USEFUL TO USE SUCH A SURFACE.



IN OTHER WORDS, THE ELECTRIC FIELD IS EVERYWHERE NORMAL TO A SPHERICAL SURFACE:

$$E_n = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

(STEP 2)

THE ELECTRIC FLUX THROUGH A (SPHERICAL) SURFACE:

$$\begin{aligned}\Phi &= \oint_S da\, E_n \\ &= \oint_S da \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right) \\ &= \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)\oint_S da \\ &= \frac{q}{\varepsilon_0} \qquad \left(\oint_S da = 4\pi r^2\right)\end{aligned}$$

NOTE: THE FLUX THROUGH THE SURFACE IS INDEPENDENT OF r!

THE FLUX THROUGH A (SPHERICAL) SURFACE CONTAINING A POINT CHARGE:

$$\Phi = \frac{q}{\varepsilon_0}$$

SINCE THE FLUX THROUGH A SURFACE CONTAINING A POINT CHARGE IS INDEPENDENT OF ITS SHAPE, THIS RESULT HOLDS FOR ANY SURFACE.

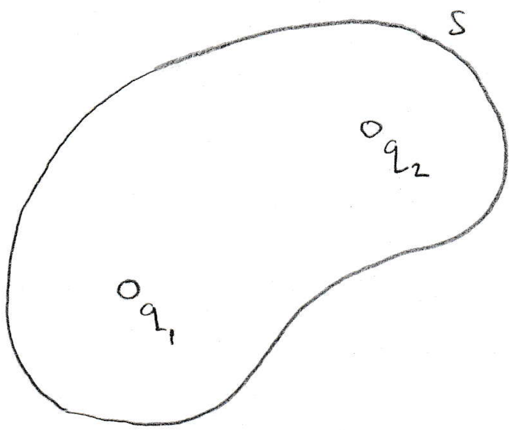
---

* CONCLUSIONS, FOR A POINT CHARGE:

$$\underset{\text{(ANY SURFACE)}}{\oint_S da\, E_n} = \begin{cases} 0 & q \text{ OUTSIDE } S \\ \dfrac{q}{\varepsilon_0} & q \text{ INSIDE } S \end{cases}$$

DRAWING #6: THE ELECTRIC FLUX THROUGH A SURFACE CONTAINING MULTIPLE POINT CHARGES I

SUPPOSE NOW THAT WE WANT TO FIND THE ELECTRIC FLUX THROUGH A SURFACE CONTAINING MULTIPLE POINT CHARGES:



THE ELECTRIC FLUX IS DEFINED PRECISELY AS BEFORE:

$$\Phi = \oint_S da\, E_n$$

DUE TO THE SUPERPOSITION PRINCIPLE, WE CAN WRITE:

$$E_n = E_{n,1} + E_{n,2}$$

HENCE:

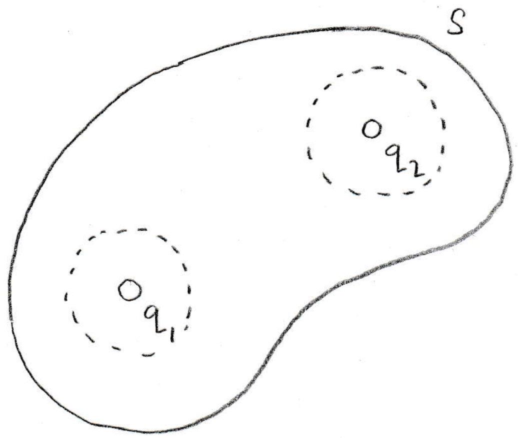
$$\begin{aligned}\Phi &= \oint_S da\, E_n \\ &= \oint_S da \left(E_{n,1} + E_{n,2}\right) \\ &= \oint_S da\, E_{n,1} + \oint_S da\, E_{n,2} \\ &= \Phi_1 + \Phi_2\end{aligned}$$

THE FLUX THROUGH A SURFACE CONTAINING MULTIPLE POINT CHARGES IS THE SUM OF THE FLUX DUE TO EACH CHARGE.

(THE SUPERPOSITION PRINCIPLE HOLDS)

DRAWING #7: THE ELECTRIC FLUX THROUGH A SURFACE CONTAINING MULTIPLE POINT CHARGES 2

WE CAN CALCULATE THE FLUX DUE TO EACH CHARGE BY SURROUNDING EACH WITH GAUSSIAN SURFACES, NOT CONTAINING THE OTHER CHARGE...



... AND CALCULATE THE FLUX DUE TO EACH CHARGE USING THE CONCLUSIONS THAT WE ALREADY MADE FOR A SINGLE POINT CHARGE.

DRAWING #8: GAUSS'S LAW

THE PRIOR RESULTS CAN BE EXTENDED TO ANY NUMBER OF POINT CHARGES OR CHARGE DENSITY:

$$\oint_S da\, E_n = \frac{Q_{INT}}{\varepsilon_0}$$

WHERE: $Q_{INT} = \sum_{i,\ \text{INSIDE } S} q_i$

$$Q_{INT} = \int_{V,\ \text{INSIDE } S} dV\, \rho \qquad (\rho\text{: CHARGE DENSITY})$$

THIS RESULT IS KNOWN AS GAUSS'S LAW IN INTEGRAL FORM.

IT IS USEFUL TO ALSO WRITE GAUSS'S LAW IN DIFFERENTIAL FORM:

FROM VECTOR CALCULUS:

(NOTE: VECTOR CALCULUS IS THE DIFFERENTIATION AND INTEGRATION OF VECTOR FIELDS.)

$$\oint_S da\, \mathbb{A}\cdot\hat{n} = \nabla\cdot\mathbb{A}\, dV$$

WHERE: $\nabla\cdot\mathbb{A} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$ (THE DIVERGENCE OF $\mathbb{A}$)

THIS SAYS THAT THE DIVERGENCE OF $\mathbb{A}$ REPRESENTS THE VOLUME DENSITY OF THE OUTWARD FLUX OF $\mathbb{A}$ FROM AN INFITESIMAL VOLUME AROUND A GIVEN POINT.

DRAWING #9: GAUSS'S LAW 2

GAUSS'S LAW IN INTEGRAL FORM CAN BE REWRITTEN IN DIFFERENTIAL FORM AS FOLLOWS:

$$\oint_S da\, \mathbb{E} \cdot \hat{n} = \frac{Q_{INT}}{\varepsilon_0}$$

$$\nabla \cdot \mathbb{E}\, dV = \frac{Q_{INT}}{\varepsilon_0}$$

THE CHARGE INSIDE AN INFITESIMAL VOLUME IS GIVEN BY:

$$Q_{INT} = \rho\, dV$$ (BY THE DEFINITION OF ρ)

THEREFORE:

$$\nabla \cdot \mathbb{E}\, dV = \frac{\rho\, dV}{\varepsilon_0}$$

$$\nabla \cdot \mathbb{E} = \frac{\rho}{\varepsilon_0}$$ (GAUSS'S LAW IN DIFFERENTIAL FORM)

NOTE: THIS EQUATION IS ONE OF THE TWO FUNDAMENTAL FIELD EQUATIONS OF ELECTROSTATICS; AND ONE OF THE FOUR FUNDAMENTAL FIELD EQUATIONS OF ELECTROMAGNETICS (MAXWELL'S EQUATIONS).

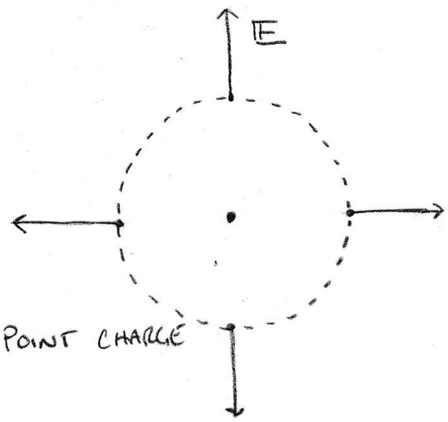
NOTE: GAUSS'S LAW IS ESSENTIALLY COULOMB'S LAW, WRITTEN IN A DIFFERENT FORM (e.g., WE COULD EASILY WORK BACKWARDS AND DERIVE IT); THOUGH, IT IS MORE FUNDAMENTAL.

NOTE: GAUSS'S LAW BY ITSELF CANNOT JUST GIVE THE SOLUTION TO ANY ELECTROSTATICS PROBLEM BECAUSE THE OTHER FUNDAMENTAL FIELD EQUATION MUST BE OBEYED TOO...

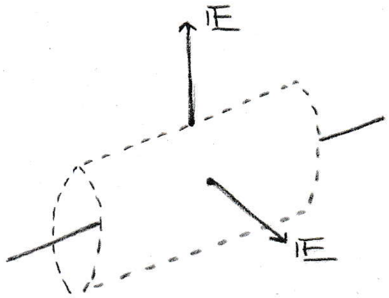
... GAUSS'S LAW IS INCREDIBLY USEFUL IF WE SUPPLEMENT IT WITH ADDITIONAL INFORMATION --- e.g., HOW THE FIELD LOOKS.

DRAWING #10: SYMMETRY

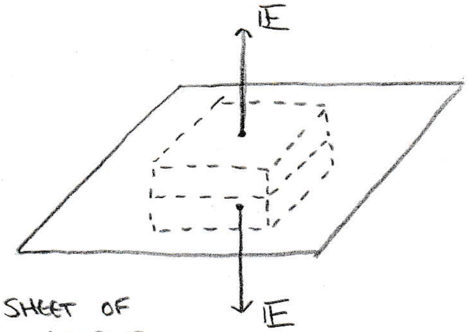
THE SYMMETRY OF THE ELECTRIC FIELD MUST MATCH THAT OF THE CHARGE DISTRIBUTION!



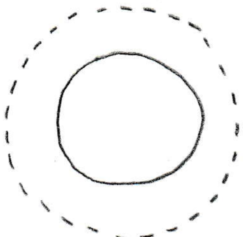
POINT CHARGE



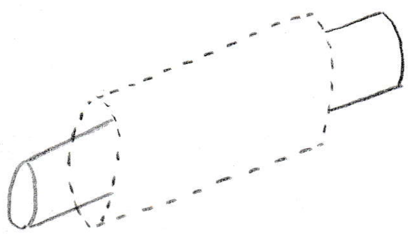
LINE CHARGE



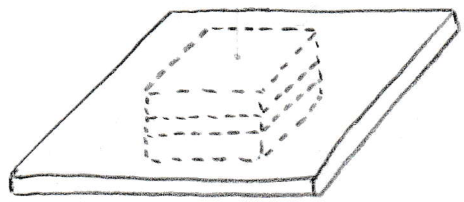
SHEET OF CHARGE



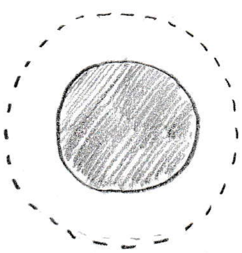
CONDUCTING SPHERE



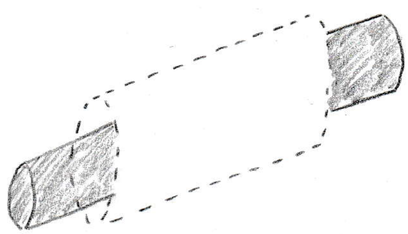
CONDUCTING CYLINDER



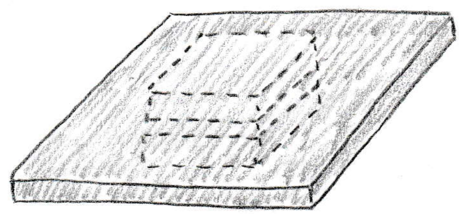
CONDUCTING SLAB



SPHERE WITH UNIFORM CHARGE DENSITY



CYLINDER WITH UNIFORM CHARGE DENSITY



SLAB WITH UNIFORM CHARGE DENSITY

SPHERICAL SYMMETRY

CYLINDRICAL SYMMETRY

PLANAR SYMMETRY

FOR THESE SITUATIONS OF SUFFICIENTLY HIGH SYMMETRY, WE CAN DRAW GAUSSIAN SURFACES OVER WHICH THE ELECTRIC FIELD IS EVERYWHERE NORMAL AND WHICH WE CAN CALCULATE THE SURFACE INTEGRAL OVER THE AREA (THE AREA) RATHER EASILY. --- i.e., IT IS WELL-SUITED FOR GAUSS'S LAW.

NOTE: A SURFACE THAT DOES NOT MATCH THE CHARGE DISTRIBUTION IS NOT TOO USEFUL.