Numerical Transients

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→ So far we have considered 0-D & 1-D analytical transients. 0-D were "lumped capacitance" with very fast internal temperature response. 1-D is initially semi-infinite until the thermal wave crosses the material extent @ which point the problem requires a seperation of variables solution & very simple requirements.

→ In many cases analytical solutions, though accurate & fast, will be overwhelmingly complicated & inflexible → where we have no choice but numerical solutions.

→ Numerical Transient problems use a combination of our 1-D transient techniques:

1) Distribute nodes across the domain
2) Energy balance on each control volume will lead to a 'state equation' for each node: $\frac{dT_i}{dt} = f(T_i \text{ for } i = 1..N, t)$
3) Numerically integrate the state equations forward in time using one of several techniques (methods) (Euler's, Heun's, Crank-Nicolson, RK4, etc)

→ State equation: provides the time rate of change of the state variable given its value & time. (e.g. temperature vs time)

We can't just construct a finite difference network through time. We have to consider trends (one point in time to predict the next time.

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Example: Most basic transient where a block of material initially in equilibrium is exposed to a step change in environmental temperature.

Step 0: Quick resistance calcs $\rightarrow R_{tot} = \frac{R_{conv,internal}}{R_{S\ to\ surroundings}} = \frac{\frac{th}{A_s k}}{\frac{1}{hA_s}}$ $\quad A_s = L_c^2$

$Bi = \frac{h L_c}{k}$ if $Bi < 0.1$ then "lumped" & system can be all of material

↳ else it's a 1-D problem & we need to discretize the spatial domain into small nodes & elements to move through time.

Step 1: Energy Balance: $\dot{E}_{in} = \dot{E}_{out} + \frac{dE_{stored}}{dt} \Rightarrow 0 = \dot{Q}_{conv} + \frac{dU}{dt}$

Step 2: Rate Equations: $\dot{Q}_{conv} = hA_s(T - T_\infty)$ & $\frac{dU}{dt} = \rho \forall C \frac{dT}{dt}$

↳ $0 = hA_s(T - T_\infty) + \rho \forall C \frac{dT}{dt} \Rightarrow \frac{dT}{dt} = -\frac{hA_s(T - T_\infty)}{\rho \forall C} \Rightarrow \frac{dT}{dt} = -\frac{(T - T_\infty)}{\tau_{lumped}}$ — State Equation

* up until this point our analytical & numerical approaches are usually the same!

Step 3: ~~Discretize~~ Define the domain by dividing the simulation time (t_{sim} or space) into time steps: $t_j = \frac{(j-1)}{(M-1)} t_{sim}$ for $j = 1 \ldots M$ & $\Delta t = \frac{t_{sim}}{(M-1)}$

Step 4: Apply a Numerical Integration Technique

↳ There are many to choose from, We'll start with the most basic & build in sophistication.

Explicit Techniques

Euler's Method: Computes the temperature @ the end of each time step based on the temperature @ the beginning of the step & the state equation. Euler's method assumes the rate of temperature change is constant within the time step & equal to the value at the beginning of the step.

$$T_{j+1} = T_j + \left.\frac{dT}{dt}\right|_{T=T_j,\, t=t_j} \Delta t$$

- T_{j+1}: Temperature @ end of time step
- T_j: Temperature @ beginning of step
- $\left.\frac{dT}{dt}\right|_{T=T_j,\, t=t_j} \Delta t$: State equation assuming rate change @ beginning is constant throughout Δt

Because T_{j+1} can be calculated explicitly using information @ the beginning, Euler's method is known as an explicit technique.

$$T_2 = T_1 - \frac{(T_1 - T_\infty)}{\tau_{lumped}} \Delta t \quad \ldots$$

Euler's Method, like all numerical techniques is not precise. If Δt is too large for the problem the technique becomes unstable. Show Figure 5-5 pg 436.

→ rearranging: $T_{j+1} = T_j \left[1 - \frac{\Delta t}{\tau_{lumped}}\right] + T_\infty \frac{\Delta t}{\tau_{lumped}}$ → this becomes unstable when this coefficient goes negative.

Thus a critical time step can be defined where $\Delta t_{crit} = \tau_{lumped}$ so that $\Delta t \ll \Delta t_{crit}$ for better accuracy.

Predictor Corrector Techniques: Heun's Method & Runga-Kutta 4th Ordered (RK-4)

Heun's method uses Euler's method to predict the temperature @ the conclusion of a time step (T_{j+1}): $T_{j+1} = T_j + \left.\frac{dT}{dt}\right|_{T=T_j,\, t=t_j} \Delta t$

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The temperature (predicted) @ the end of the step $\hat{T}_{j+1}$ then predicts the rate of temperature @ the end of the step $\left(\frac{dT}{dt}\Big|_{T=\hat{T}_{j+1},\, t=t_{j+1}}\right)$. Then a 'corrector' step averages the rates of change @ the beginning & end of the step to predict T_{j+1} actual.

$$T_{j+1} = T_j + \left[\frac{dT}{dt}\Big|_{T=T_j,\, t=t_j} + \frac{dT}{dt}\Big|_{T=\hat{T}_{j+1},\, t=t_{j+1}}\right]\frac{\Delta t}{2}$$

RK estimate $\frac{dT}{dt}$ 4 times & averages for each time step.

Implicit Techniques: Fully Implicit, Crank-Nicolson, & EES INTEGRAL functions require an implicit solver but avoid the critical timestep issue.

Fully Implicit: $T_{j+1} = T_j + \frac{dT}{dt}\Big|_{T=T_{j+1},\, t=t_{j+1}} \Delta t$ & $T_{j+1} = T_j - \frac{(T_{j+1} - T_\infty)}{\tilde{\tau}_{lumped}}\Delta t$

Show Figure 3-7 & not stability.

Crank-Nicolson: Combines Euler's method & fully implicit method.

$$T_{j+1} = T_j + \left[\frac{dT}{dt}\Big|_{T=T_j,\, t=t_j} + \frac{dT}{dt}\Big|_{T=\hat{T}_{j+1},\, t=t_{j+1}}\right]\frac{\Delta t}{2} \quad \& \quad T_{j+1} = T_j - \left[\frac{(T_j - T_\infty)}{\tilde{\tau}_{lumped}} + \frac{(T_{j+1} - T_\infty)}{\tilde{\tau}_{lumped}}\right]\frac{\Delta t}{2}$$

Combines high accuracy & stability.

EES INTEGRAL: Third Order semi-implicit with adaptive step-size (Way cool & complex)

Show Accuracy plot of all techiques.