# Analysis of microwave heating for cylindrical shaped objects

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## ARTICLE INFO

## ABSTRACT

Microwave heating is very popular in food industries as well as in home and office to warm up foodstuffs quickly. However, this technique provides non-uniform heating within the system. The potential existence of standing wave due to the transmission and reflection from interfaces is responsible for this non-uniform heating. Therefore, it is important to study the coupling between electromagnetic wave propagation and energy transfer in the system to predict the temperature distribution within the foodstuff. In this paper, a closed-form analytic solution is presented to predict the temperature distribution within a cylindrical shaped foodstuff under microwave heating by solving an unsteady energy equation. A simplified Maxwell's equation is solved for electric field distribution within the body. The heat generation by microwave is calculated from the electric field distribution within the body using Poynting theorem. The effects of cylinder radius, heat transfer coefficient and incident frequency are studied for different length of the cylindrical foodstuff. It is found that the temperature within the body is very sensitive to cylinder length and time. The results indicate that uniform and effective heating depend on the proper integration of geometric parameters and dielectric properties of the object as well as the frequency of the incident electromagnetic wave. This rigorous analytic investigation will provide significant insight to understand and overcome the challenge of non-uniformity in temperature distribution in microwave heating.

## 1. Introduction

Microwave heating is widely used to warm up foodstuffs quickly. Unlike conventional heating, where heat flows from the outer surface to the core of the body, heat is generated within the body in microwave heating due to the transmission of an electromagnetic wave, and heat can propagate in all directions depending on temperature. From energy point of view, microwave heating is very efficient method as it does not unnecessarily warm up the space surrounding the object of interest. Because of its pollution free, rapid heating technique, it has become increasingly popular in other industries too [1].

In microwave heating, the modulation in electromagnetic field polarizes the molecules in dielectric materials and creates dipole moments that cause these molecules to rotate. The molecular friction resulting from this dipolar rotation of polar solvent causes heat generation in the body [2]. Due to this intrinsic heat generation capability, microwave heating can provide prompt rise of temperature within the low thermal conductive product, especially in

food items. However, this technique is blamed for uneven heating as the food product processed with microwave technique shows alternate hot and cold spots [3].

There are a number of adverse consequences of uneven heating of foodstuff. First, the uneven heating is responsible for poor food quality such as overheating or burning of part of the foodstuff where the temperature is the highest. Second, the food texture is affected by the non-uniform temperature within the food. Sometime it is even difficult to recognize the original texture of the food if the temperature difference between consecutive hot and cold spots is extreme. Third, harmful microorganisms such as bacteria may remain active in the food due to the presence of cold spots. The presence of live microorganisms in the foodstuff is a major concern for human health and safety as pathogenic bacteria can contribute to food borne diseases. A recent study has shown that non-uniform temperature distribution within chickens facilitates the pathogenic growth at faster pace [4].

The occurrence of non-uniform heating is the major limiting factor for widespread application of microwave based heating technique. The uneven heating is primarily caused by the non-uniform distribution of microwave energy in the foodstuff due to factors such as dielectric loss, penetration depth, thickness, shape and size of the product [5]. In an experimental study, Sakai and Wang [6] confirmed the influence of dielectric properties of material on

**Nomenclature**

| | | | |
|---|---|---|---|
| $A$ | coefficient [V/m] | $\beta$ | attenuation factor [rad/m] |
| $B$ | coefficient [V/m] | $\chi$ | propagation constant [$\mathrm{m^{-1}}$] |
| $\vec{B}$ | magnetic induction [Wb/m$^2$] | $\delta$ | reflection phase angle [rad] |
| $C$ | velocity of light [m/s] | $\varepsilon$ | permittivity [F/m] |
| $c_P$ | specific heat capacity [J/kg K] | $\varepsilon_0$ | free-space permittivity [F/m] |
| $\vec{D}$ | electric displacement [C/m$^2$] | $\gamma$ | ratio of heat transfer coefficient and thermal |
| $\vec{E}$ | electric field [V/m] | | conductivity [$\mathrm{m^{-1}}$] |
| $E_0$ | incident electric field [V/m] | $\Gamma$ | decomposed temperature in $t$ |
| $f$ | frequency [MHz] | $\phi$ | azimuthal direction [rad] |
| $\vec{H}$ | magnetic field intensity [A/m] | $\zeta$ | impedance [ohm] |
| $h$ | heat transfer coefficient [W/m$^2$ K] | $\theta$ | modified temperature [°C] |
| $I$ | incident microwave energy flux [W/cm$^2$] | $\kappa'$ | relative dielectric constant |
| $\vec{J}$ | current density [A/m$^2$] | $\kappa''$ | relative dielectric loss |
| $K$ | kernel | $\lambda$ | eigenvalues |
| $k$ | thermal conductivity [W/m K] | $\eta$ | eigenvalues |
| $L$ | length of the food cylinder [m] | $\mu$ | permeability [H/m] |
| $N$ | normalization integral | $\mu_0$ | free-space permeability [H/m] |
| $\vec{n}$ | unit normal vector | $\rho$ | density [kg/m$^3$] |
| $Q$ | power generation [W/m$^3$] | $\rho_v$ | charge density [C/m$^3$] |
| $\vec{q}$ | Poynting vector [W/m$^2$] | $\sigma$ | phase factor [rad/m] |
| $R_{01}$ | reflection coefficient | $\sigma'$ | electrical conductivity [S/m] |
| $R$ | decomposed temperature in $r$ | $\xi$ | dimensionless length |
| $r$ | radial direction | $\omega$ | angular frequency [rad/s] |
| $r_o$ | radius of food cylinder [m] | | |
| $T$ | temperature [°C] | *Subscripts* | |
| $t$ | time [s] | $i$ | initial |
| $T_{01}$ | transmission coefficient | $l$ | material index |
| $Z$ | decomposed temperature in $z$ | $m$ | eigenvalue index |
| $z$ | axial direction | $n$ | eigenvalue index |
| | | $\infty$ | surrounding |
| *Greek symbols* | | 0 | free space |
| $\alpha$ | thermal diffusivity [m$^2$/s] | 1 | within food cylinder |

the temperature distribution. Gunasekaran and Yang [7] studied the effect of parameters such as size, geometry, pulsating ratio, and microwave processing time on sample temperature distribution and concluded that the pulsed microwave heating provides more uniform heating than continuous microwave heating. Funawatashi and Suzuki [8] classified the reasons for non-uniform heating into two categories based on the size of the object. For smaller size foodstuff, standing wave is dominant as interaction between transmission and reflection waves takes place within the object. On the other hand, for larger size object the rapid decay of incident wave is the primary cause for the non-uniform heating.

There exist a number of numerical works to study the temperature distribution due to microwave heating. Ayappa et al. [9,10] reported the temperature distribution in food slabs and cylindrical shaped food materials and suggested avoiding corners and edges to reduce the localized rapid heating of the samples. Yang and Gunasekaran [11] predicted the temperature distribution in 2% agar gel cylinder and reported that the Poynting theorem is much more accurate than that of Lambert's law for calculating heat generation, especially in the boundary region. Role of different parameters such as geometry and composition, dielectric parameters, mode of incident microwaves are also extensively studied [12–15].

Though numerous experimental and numerical works on microwave heating were reported, very little attention has been paid to predict temperature distribution in 2D or 3D foodstuff using analytic technique. Fleischman [16] obtained an analytic expression for predicting temperature in 1D beef slab by solving Maxwell's and energy equations. In this study, a closed-form analytic solution is obtained to predict the temperature distribution in 3D cylindrical shaped food objects. This analytic solution can be

used to design foodstuff that requires microwave heating. The rest of the paper is organized as follows: First the theory and governing equations for microwave heating are provided. Then an analysis to find the microwave power generation from electromagnetic wave is presented. Next, the temperature distribution within the body is obtained from the energy equation using integral transform technique. This is followed by a discussion on microwave heating of cylindrical shaped objects of different lengths for various radius, heat transfer coefficients and incident frequencies. Finally, we present our conclusions on this analytic work.

## 2. Theory

Microwave heating takes place when an electromagnetic wave with a frequency between 300 MHz and 300 GHz [14] passes through a media to generate heat in the system. The heat generation is primarily due to the molecular friction of dielectric molecules such as water and fat in the system when an electromagnetic field is applied. The dielectric molecules are electric dipoles meaning that they have a positive charge at one end and a negative charge at the other. Hence, these diploes rotate as they try to align themselves with the alternating electric field of the microwaves. Therefore, to study the microwave heating one has to understand the electromagnetic field within the system as well as its effect on energy equation.

There are two approaches for calculating microwave power generation based on the size of the foodstuff: Poynting theorem and Lambert's law. Olievera and Franca [2] reported a comparative study of Poynting theorem and Lambert law for microwave heating

of food slabs and cylinders. Ayappa et al. [9] showed that the heat generated by microwave can be predicted well from Poynting theorem for small size foodstuff whereas Lambert law can be used for bulky, thick sample size. The microwave power formulation by Lambert's law neglects all reflection, and hence this approach is valid only for semi-infinite samples [17] and cannot be used for smaller food items such as nuggets. In this study, we adopted Poynting theorem to calculate the microwave heat generation in the foodstuff from the electromagnetic field distribution due to the finite size of the cylindrical foodstuff.

### 2.1. Governing equations

The microwave electromagnetic field distribution within the material is governed by Maxwell's equations as [18]

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1a}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{1b}$$

$$\vec{\nabla} \cdot \vec{D} = \rho_v \tag{1c}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1d}$$

where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic induction, $\vec{H}$ is the magnetic field, $\vec{J}$ is the current density, $\vec{D}$ is the electric displacement, and $\rho_v$ is the electric charge density. The heat generated by the electromagnetic wave can be calculated from Poynting theorem as [18]

$$\vec{q} = \frac{1}{2}\vec{E} \times \vec{H}^* \tag{2a}$$

$$Q(z) = -\text{Re}(\vec{\nabla} \cdot \vec{q}) \tag{2b}$$

where $\vec{q}$ is the Poynting vector and $\vec{H}^*$ is the complex conjugate of the magnetic field. For microwave heating, the temperature distribution within the system is governed by the energy equation as [18]:

$$\rho C_p \frac{dT}{dt} = \vec{\nabla} \cdot (k \vec{\nabla} T) + Q \tag{3}$$

### 2.2. Assumptions

In this study, we considered a cylindrical shaped object (Fig. 1) which is subjected to microwave heating. For simplicity, we assume that microwaves are transverse electromagnetic or uniform plane waves. Although a uniform plane wave cannot be formed in a real system, the electric field distribution obtained from this simplified model can approximate the actual electric field in the system [18]. The other assumptions used to simplify the problem are:

   (i) Food system obeys linear material constitutive laws.
   (ii) Electroneutrality condition is satisfied within the system.
   (iii) The magnetic permeability $\mu(\omega)$ can be approximated by its value in free space.
   (iv) Dielectric properties are temperature independent.
   (v) Material properties such as thermal conductivity, specific heat are temperature independent.

## 3. Analysis

### 3.1. Microwave radiation and power formulation

Maxwell's equations can be simplified by using the assumptions mentioned above and by applying following material constitutive relations [18]
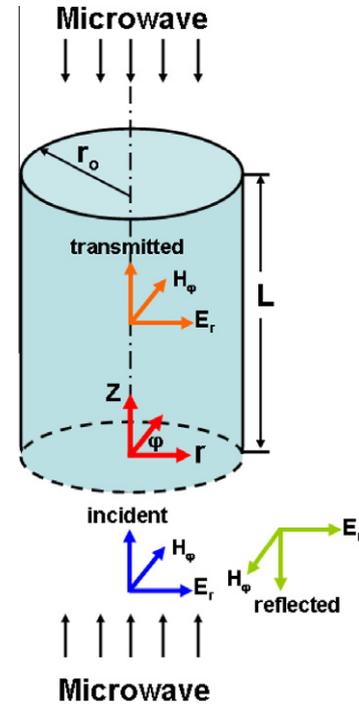


**Fig. 1.** Schematic of a cylindrical shaped foodstuff heated by microwave. In a uniform plane wave, the magnetic field is always perpendicular to the electric field.

$$\vec{J} = \sigma' \vec{E} \tag{4a}$$

$$\vec{D} = \varepsilon \vec{E} \tag{4b}$$

$$\vec{B} = \mu \vec{H} \tag{4c}$$

where $\sigma'$ is the electrical conductivity, $\varepsilon$ is the dielectric constant, and $\mu$ is the magnetic permeability. If microwave incident rays are propagating in the axial direction, then

$$E_z = E_\phi = 0 \tag{5a}$$

$$\frac{\partial E_r}{\partial r} = \frac{\partial E_r}{\partial \phi} = 0 \tag{5b}$$

Therefore, for a uniform plane wave (Fig. 1), the simplified equation for electric field can be written as:

$$\frac{d^2 E_r}{dz^2} + \chi^2 E_r = 0 \tag{6}$$

where $\chi$ is the propagation constant, and it can be given as $\chi^2 = \omega^2 \mu_0 \varepsilon_0 (\kappa' + i\kappa'')$. The propagation constant can also be expressed as $\chi = \sigma + i\beta$, where the phase factor [9],

$$\sigma = \frac{2\pi f}{c} \sqrt{\frac{\kappa'(\sqrt{1 + \tan^2 \delta} + 1)}{2}} \tag{7a}$$

and the attenuation factor [9],

$$\beta = \frac{2\pi f}{c} \sqrt{\frac{\kappa'(\sqrt{1 + \tan^2 \delta} - 1)}{2}} \tag{7b}$$

For cylindrical object (Fig. 1) with incident microwave radiations from the top and bottom surfaces, the boundary conditions for uniform plane microwave propagating in $z$ direction can be given as

$$E_{r,0}\big|_{z=0,L} = E_{r,1}\big|_{z=0,L} \tag{8a}$$

$$\frac{1}{\mu_0 \omega} \frac{dE_{r,0}}{dz}\bigg|_{z=0} = \frac{1}{\mu_1 \omega} \frac{dE_{r,1}}{dz}\bigg|_{z=0} \tag{8b}$$

where 0 and 1 indicates the free and food space, respectively. The solution of the simplified Maxwell's equation (Eq. (6)) is given by

$$E_{r,l} = A_l e^{i\chi_l z} + B_l e^{-i\chi_l z} \tag{9}$$

where subscript $l$ ($l = 0$: air; $l = 1$: food) represents the media through which microwave propagates. Applying the boundary conditions (Eqs. (8a) and (8b)) in the general solution (Eq. (9)), the coefficient of $A_l$ and $B_l$ within the food system can be found as:

$$A_1 = \frac{T_{01}E_0}{1 + R_{01}e^{i\chi_1 L}} \tag{10a}$$

and

$$B_1 = \frac{T_{01}E_0 e^{i\chi_1 L}}{1 + R_{01}e^{i\chi_1 L}} \tag{10b}$$

where the transmission coefficient,

$$T_{01} = \frac{2\zeta_1}{\zeta_1 + \zeta_0} \tag{11a}$$

the reflection coefficient,

$$R_{01} = \frac{\zeta_1 - \zeta_0}{\zeta_1 + \zeta_0} \tag{11b}$$

and the intrinsic impedance, $\zeta = \frac{\mu\omega}{\chi}$ [18]. Since electric field propagates along the axial direction only, the subscripts will be dropped from now on, and the electric field distribution in the food system can be expressed as:

$$E = \frac{T_{01}E_0}{1 + R_{01}e^{i\chi_1 L}}(e^{i\chi_1 z} + e^{i\chi_1 (L-z)}) \tag{12}$$

Once the electric field distribution is known, the magnetic field distribution can be evaluated eventually as they are related by,

$$\frac{dE_{r,l}}{dz} = i\mu_l \omega H_{\varphi,l} \tag{13}$$

Applying Poynting power theorem, the power dissipated per unit volume is given by:

$$Q(z) = \frac{1}{2}\omega\varepsilon_0\kappa''|E|^2 \tag{14}$$

Using Eq. (12), the volumetric power generation within the food slab is obtained as:

$$Q(z) = \frac{1}{2}\omega\varepsilon_0\kappa''|E_0|^2|T_{01}|^2$$
$$\times \frac{e^{-2\beta z} + e^{-2\beta(L-z)} + 2e^{-\beta L}\cos(2\sigma z - \sigma L)}{1 + 2|R_{01}|e^{-\beta L}\cos(\delta_{01} + \sigma L) + |R_{01}|^2 e^{-2\beta L}} \tag{15}$$

The absolute value of reflection coefficient and transmission coefficient can be expressed as a function of the dielectric properties as [17]

$$|R_{01}| = \left|\frac{1 - \sqrt{\kappa' - i\kappa''}}{1 + \sqrt{\kappa' - i\kappa''}}\right| \tag{16a}$$

$$|T_{01}| = \left|\frac{2}{1 + \sqrt{\kappa' - i\kappa''}}\right| \tag{16b}$$

### 3.2. Heat equation and temperature distribution

The energy equation for a cylindrical shaped object (Fig. 1) under microwave heating can be simplified as

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{Q(z)}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \text{in } 0 \leqslant r < r_0; \ 0 < z < L \tag{17}$$

where $\alpha$ is the thermal diffusivity, $k$ is the thermal conductivity, and $T$ is the temperature. The expression of volumetric heat source $Q(z)$

is presented in Eq. (15). For our problem, the initial and boundary conditions for energy equation are:

$$T = T_i \quad \text{at } t = 0 \tag{18a}$$

$$T = finite \quad \text{at } r = 0; \ t > 0 \tag{18b}$$

$$-k\frac{\partial T}{\partial r} = h(T - T_\infty) \quad \text{at } r = r_0; \ t > 0 \tag{18c}$$

$$k\frac{\partial T}{\partial z} = h(T - T_\infty) \quad \text{at } z = 0; \ t > 0 \tag{18d}$$

$$-k\frac{\partial T}{\partial z} = h(T - T_\infty) \quad \text{at } z = L; \ t > 0 \tag{18e}$$

An analytic expression for temperature distribution can be obtained from the above second order, nonhomogenous partial differential equation using integral transform technique. To apply integral transform technique, the governing equation is simplified to

$$\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{\partial z^2} + \frac{Q(z)}{k} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \quad \text{in } 0 \leqslant r < r_0; \ 0 < z < L \tag{19}$$

where $\theta(r,z,t) = T - T_\infty$. Therefore, the initial and boundary conditions for this modified governing equation are reduced to the following:

$$\theta = \theta_i \quad \text{at } t = 0 \tag{20a}$$

$$\theta = finite \quad \text{at } r = 0; \ t > 0 \tag{20b}$$

$$\frac{\partial \theta}{\partial r} + \gamma\theta = 0 \quad \text{at } r = r_0; \ t > 0 \tag{20c}$$

$$-\frac{\partial \theta}{\partial z} + \gamma\theta = 0 \quad \text{at } z = 0; \ t > 0 \tag{20d}$$

$$\frac{\partial \theta}{\partial z} + \gamma\theta = 0 \quad \text{at } z = L; \ t > 0 \tag{20e}$$

where $\theta_i = T_i - T_\infty$ and $\gamma = \frac{h}{k}$. The partial derivative with respect to $r$ can be eliminated from energy equation (19) by introducing Hankel transform as [19]

$$\bar{\theta}(\lambda_m, z, t) = \int_0^{r_0} K(\lambda_m, r)\theta(r, z, t)r dr \tag{21}$$

The kernels $K(\lambda_m, r)$ are the normalized characteristic functions of the following eigenvalue problem:

$$r^2\frac{d^2 R}{dr^2} + r\frac{dR}{dr} + \lambda^2 r^2 R^2 = 0 \tag{22}$$

$$R = finite \quad \text{at } r = 0; \ t > 0 \tag{23a}$$

$$\frac{dR}{dr} + \gamma R = 0 \quad \text{at } r = r_0; \ t > 0 \tag{23b}$$

where $\lambda$ is the eigenvalue. For the above eigenvalue problem, the characteristic function is $J_0(\lambda_m r)$, the normalization integral is $N_m = \int_0^{r_0} r J_0^2(\lambda_m r)dr$, and the kernel is $K(\lambda_m, r) = \frac{J_0(\lambda_m r)}{\sqrt{N_m}}$. The eigenvalues ($\lambda_m$) are the positive roots of

$$-\lambda_m J_1(\lambda_m r_o) + \gamma J_0(\lambda_m r_o) = 0 \tag{24}$$

The transformation of energy equation (19) with respect to $r$, through the use of Eq. (21) yields

$$-\lambda_m^2\bar{\theta} + \frac{\partial^2\bar{\theta}}{\partial z^2} + \frac{\overline{Q}(z)}{k} = \frac{1}{\alpha}\frac{\partial\bar{\theta}}{\partial t} \tag{25}$$

where $\overline{Q}(z) = Q(z)\frac{r_o}{\lambda_m}J_1(\lambda_m r_o)$. Our next goal is to remove the partial derivate with respect to $z$ from the transformed energy equation (25) by introducing Fourier transform [19] as

$$\bar{\bar{\theta}}(\lambda_m, \eta_n, t) = \int_0^L K(\eta_n, z)\bar{\theta}(\lambda_m, z, t)dz \tag{26}$$

Here the kernels $K(\eta_n, z)$ are the normalized characteristic functions of the following eigenvalue problem:

$$\frac{d^2Z}{dz^2} + \eta^2 Z^2 = 0 \qquad (27)$$

$$-\frac{dZ}{dz} + \gamma Z = 0 \quad \text{at } z = 0; \ t > 0 \qquad (28a)$$

$$\frac{dZ}{dz} + \gamma Z = 0 \quad \text{at } z = L; \ t > 0 \qquad (28b)$$

where $\eta$ is the eigenvalue. For $z$-directional eigenvalue problem, the kernel can be found as $K(\eta_n, z) = \frac{\sqrt{2}(\eta_n \cos \eta_n z + \gamma \sin \eta_n z)}{\sqrt{L(\eta_n^2 + \gamma^2) + 2\gamma}}$. The eigenvalues $\eta_n$ are the positive roots of

$$\tan(\eta_n L) = \frac{2\gamma \eta_n}{\eta_n^2 - \gamma^2} \qquad (29)$$

Now the transformation of Eq. (25) with respect to $z$, through the use of Fourier transform (Eq. (26)), yields

$$-\alpha(\lambda_m^2 + \eta_n^2)\bar{\bar{\theta}} + \Phi = \frac{d\bar{\bar{\theta}}}{dt} \qquad (30)$$

where

$$\Phi = \frac{\alpha \bar{\bar{Q}}(\eta_n)}{k} = \frac{\alpha r_o}{k\lambda_m} J_1(\lambda_m r_o) \int_0^L Q(z)(\eta_n \cos \eta_n z + \gamma \sin \eta_n z)dz \qquad (31)$$

The second transformation gives us an ordinary differential equation (Eq. (30)), and it can be solved readily as

$$\bar{\bar{\theta}}(\lambda_m, \eta_n, t) = \frac{\Phi}{\alpha(\lambda_m^2 + \eta_n^2)} + \left(\bar{\bar{\theta}}_i - \frac{\Phi}{\alpha(\lambda_m^2 + \eta_n^2)}\right)e^{-\alpha(\lambda_m^2 + \eta_n^2)t} \qquad (32)$$

where

$$\bar{\bar{\theta}}_i = \frac{\theta_i r_o}{\lambda_m} J_1(\lambda_m r_o)\left(\sin \eta_n L - \frac{\gamma}{\eta_n}\cos \eta_n L + \frac{\gamma}{\eta_n}\right) \qquad (33)$$

Our next goal is to apply appropriate inversion techniques to obtain an analytic expression for temperature. First, the inverse Fourier transform $\bar{\theta}(\lambda_m, z, t) = \sum_{n=1}^{\infty} K(\eta_n, z)\bar{\bar{\theta}}(\lambda_m, \eta_n, t)$ is used to obtain transformed temperature as

$$\bar{\theta}(\lambda_m, z, t) = \sum_{n=1}^{\infty} \frac{2(\eta_n \cos \eta_n z + \gamma \sin \eta_n z)}{L(\eta_n^2 + \gamma^2) + 2\gamma}\bar{\bar{\theta}}(\lambda_m, \eta_n, t) \qquad (34)$$

Next the inverse Hankel transform $\theta(r, z, t) = \sum_{m=1}^{\infty} K(\lambda_m, r)\bar{\theta}(\lambda_m, z, t)$ is applied to find an expression for temperature as

$$\theta(r, z, t) = T(r, z, t) - T_\infty$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} 4\frac{\lambda_m^2 J_0(\lambda_m r)(\eta_n \cos \eta_n z + \gamma \sin \eta_n z)}{[r_o^2 J_0^2(\lambda_m r_o)(\gamma^2 + \lambda_m^2)][L(\eta_n^2 + \gamma^2) + 2\gamma]}\bar{\bar{\theta}}(\lambda_m, \eta_n, t) \qquad (35)$$

## 4. Results and discussion

Analytic expressions for electric field, power absorption, and temperature are obtained for microwave heating of cylindrical shaped food objects subjected to convective boundary conditions at all surfaces. For simplicity, we considered the transverse electromagnetic wave which propagates in the direction of incident microwave. In this study, beef is considered as the food material, though this technique can be used for any type of materials as long as the dielectric constant remains constant in the system. The dielectric properties of beef are presented in Table 1, while other properties and input parameters related to microwave heating is presented in Table 2. The incidence microwave energy flux ($I = \frac{1}{2}c\varepsilon_0 E_0^2$) was kept constant at 3 W/cm²; this value corresponds to a 1.2 kW household microwave. For brevity, the radius of food

**Table 1**
Dielectric properties of beef cylinder at different frequencies.

| Properties/frequency (MHz) | 2800 | 2450 | 915 | 300 |
|---|---|---|---|---|
| Dielectric constant, $\kappa'$ [9] | 33.6 | 30.5 | 35.4 | 38 |
| Dielectric loss, $\kappa''$ [9] | 12.60 | 9.6 | 16.0 | 47 |
| Transmission coefficient, $|T_{01}|$ | 0.2867 | 0.301 | 0.2773 | 0.2302 |
| Reflection coefficient, $|R_{01}|$ | 0.7181 | 0.7026 | 0.7291 | 0.7926 |
| Phase angle, $\delta_{01}$ (rad) | −3.08 | −3.09 | −3.07 | −3.03 |
| Phase factor, $\sigma$ (rad/m) | 345.66 | 288 | 116.76 | 44.08 |
| Attenuation factor, $\beta$ (rad/m) | 62.69 | 44.2 | 25.16 | 21.04 |

**Table 2**
Thermal properties and input parameters for analytic calculations.

| Parameters | Values |
|---|---|
| Heat transfer coefficient, $h$ (W/m²) | 1.5, 10, 25, 50 |
| Specific heat capacity, $c_p$ (J/kg K) | 3200 |
| Thermal conductivity, $k$ (W/m K) | 0.466 |
| Density, $\rho$ (kg/m³) | 1030 |
| Initial temperature, $T_i$ (°C) | 25 |
| Surrounding temperature, $T_\infty$ (°C) | 25 |
| Length of beef cylinder (cm) | 1.25, 2.50, 3.75, 5.0 |
| Incidence microwave energy flux, $I_T = I_B$ (W/cm²) | 3.00 |
| Radius of beef cylinder, $r_o$ (cm) | 0.5, 1.0 |
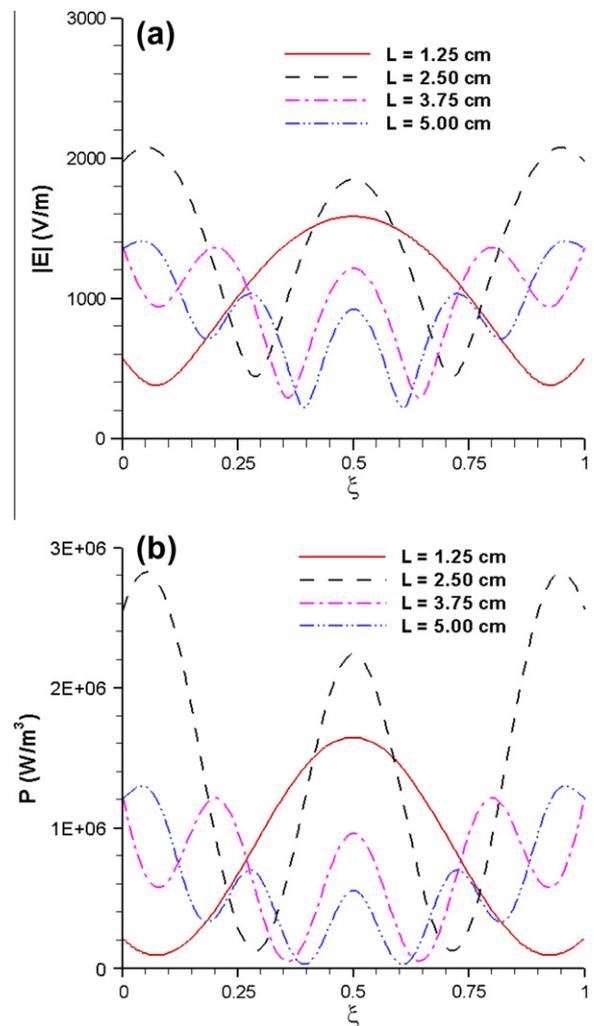


**Fig. 2.** (a) Electric field and (b) absorbed power distribution along the centerline of a cylindrical foodstuff for $f$ = 2450 MHz.

item is kept constant at 0.5 and 1.0 cm, while the length of the cylinder is varied from case to case to elucidate the effect of microwave energy in temperature distribution with time.

The electric field distribution along the axial direction ($z$) is shown in Fig. 2(a) for four different cases based on the length of the cylinder. For convenience in comparison, a normalized axial variable ($\xi = x/L$) is used in this study. Results show that the length of the food cylinder has a significant effect on the electric field distribution. For smaller cylinder the maximum electric field strength is generally found at the middle of the cylinder, while for longer cylinder the maximum field strength occurs at the both ends of the cylinder. The existence of local maxima is a result of resonance due to the interaction of transmission and reflection waves. The other parameters that might affect the electric field distribution are dielectric properties and the frequency of the incident radiation, and these effects will be discussed later.

The heat generated by the microwave closely follows the electric field distribution in the system as shown in Fig. 2b. Since we used a uniform plane wave, there is no variation in electric field or volumetric heat generation in the radial ($r$) or azimuthal ($\phi$) direction. The analytical expression of power distribution (Eq. (15)) provides the functional relationship among power, length, frequency and dielectric properties. Therefore, the information of

local maxima or minima of absorbed (microwave) power can be found based on those parameters before designing an object for microwave processing. The exact location of maxima is especially important for microwave treatment of biological tissue [20] and cancer cells [21]. Ayappa et al. [10] also reported similar observation in their experimental and numerical work.

The microwave power absorption (generation) within the system is used as a heat source in the energy equation. Hence, the power distribution (Fig. 2b) primarily dictates the microwave heating and the resultant temperature distribution. Fig. 3 shows the two dimensional temperature distributions for different length cylinders at 120 s. In all cases, the temperature is the highest along the centerline of the cylinder. Among all cases, the two dimensionality of the temperature distribution is more vivid for $L = 3.75$ cm especially at the both ends due to the presence of resonance peaks (maxima) in power absorption and thermal convection. The analytic expression also indicates that the temperature distribution in the radial direction will change significantly with convective heat transfer coefficient. The effect of convective heat transfer coefficient on the temperature distribution will be discussed later.

Fig. 3 also shows that the temperature distribution will be more like 1D (except very close to the top or bottom surface) as the channel length decreases. However, it is very interesting to note
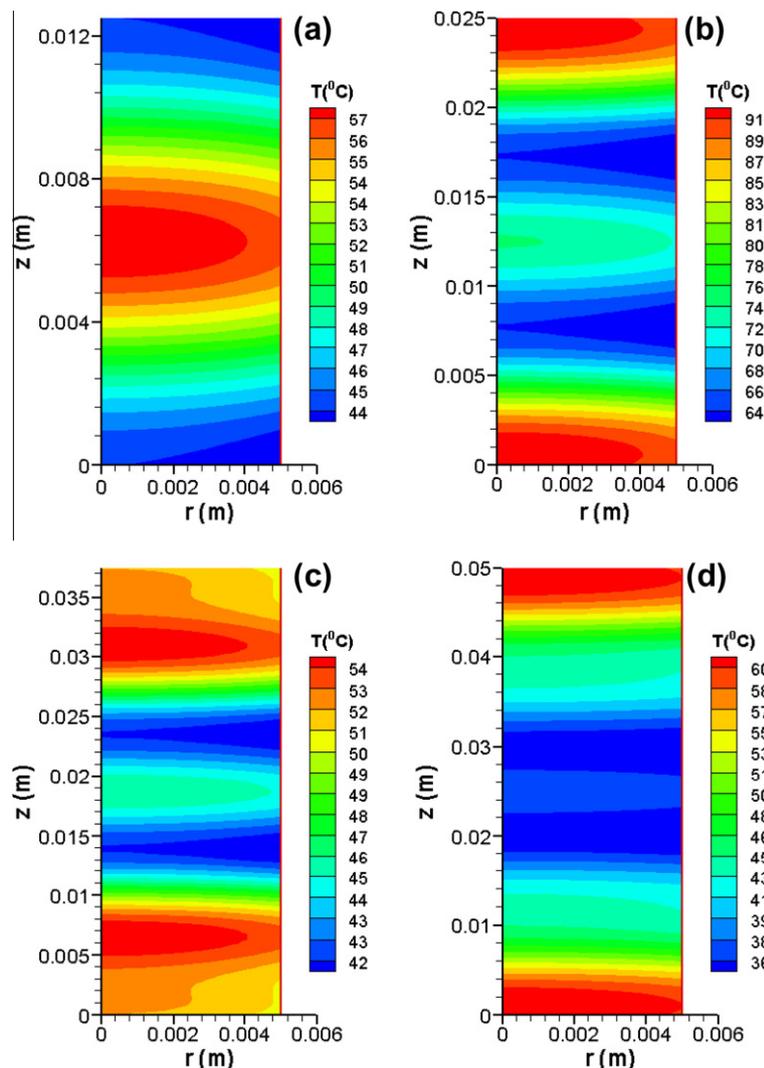


**Fig. 3.** Two dimensional temperature contour for 120 s microwave heating of a 1 cm diameter beef cylinder of length (a) $L = 1.25$ cm, (b) $L = 2.50$ cm, (c) $L = 3.75$ cm, and (d) $L = 5.0$ cm. Here $h = 10$ W/m$^2$ K and $f = 2450$ MHz.

that there is no clear trend on axial temperature distribution in the system as the length of the cylinder changes. In conventional (oven) heating system, for the same input power, initial temperature, and convective heat transfer coefficient, one would expect to have the highest temperature rise for the shortest cylinder ($L$ = 1.25 cm) and the lowest temperature increase in the longest one ($L$ = 5 cm). But for microwave heating the maximum temperature occurred for 2.5 cm long foodstuff due to the generation of strongest power peaks (Fig. 2b). The non-uniformity in the temperature distribution (Fig. 3) indicates that in microwave heating process the heat generation rate is much faster than the heat conduction rate due to the low thermal conductivity of the food item. This means that the time scale for generation is shorter than that of thermal transport.

The effect of cylinder radius on temperature distribution is also studied by keeping other parameters (frequencies, dielectric properties, heat transfer coefficient) same. The temperature contour in a 2 cm diameter beef rod is illustrated in Fig. 4 for 120 s of microwave heating. Like Fig. 3, four different cases are considered based on the length of the cylinder. Although the temperature value is little higher for 2 cm diameter cylinders, the overall trend in temperature distribution is same as 1 cm diameter cylinder. Hence, for the rest of the paper, the diameter of the cylinder is kept constant at 1 cm.

We also investigated the temperature increase with time. Fig. 5 presents the temperature distribution along the centerline of the cylinder. Results show that the temperature increases monotonically with time, and this behavior is consistent with experimental observation. One can infer that the axial variation of temperature is governed by the nature of microwave power absorption (Fig. 2b) as discussed before. For shorter cylinder length, the temperature is highest at the core and lowest on the edge. However, this trend reverses as the length of the cylinder increases. For instance, the core region is quite cold compare to the edge (Fig. 5d). With the increasing time, the difference between the minimum and maximum temperature also increases (Fig. 6a). The temperature distribution in different length cylinders indicates that there might be an optimum length for a given dielectric material and frequency where the thermodynamic efficiency of the heating system is highest. This optimum length can be estimated from the number of resonances occurred during the microwave propagation. Ayappa et al. [10] demonstrated that the number of resonances in microwave power distribution within a sample can be expressed as, $n = 2L/\varsigma_s$, where $L$ is the length of the sample and $\varsigma_s$ is the wavelength of electromagnetic wave. Therefore the optimum length for a beef cylinder heated at 2450 MHz frequency will be 2.19 cm ($n \approx 2$) where the average output power (heat generation) and efficiency become maximum (Fig. 6b). The heating
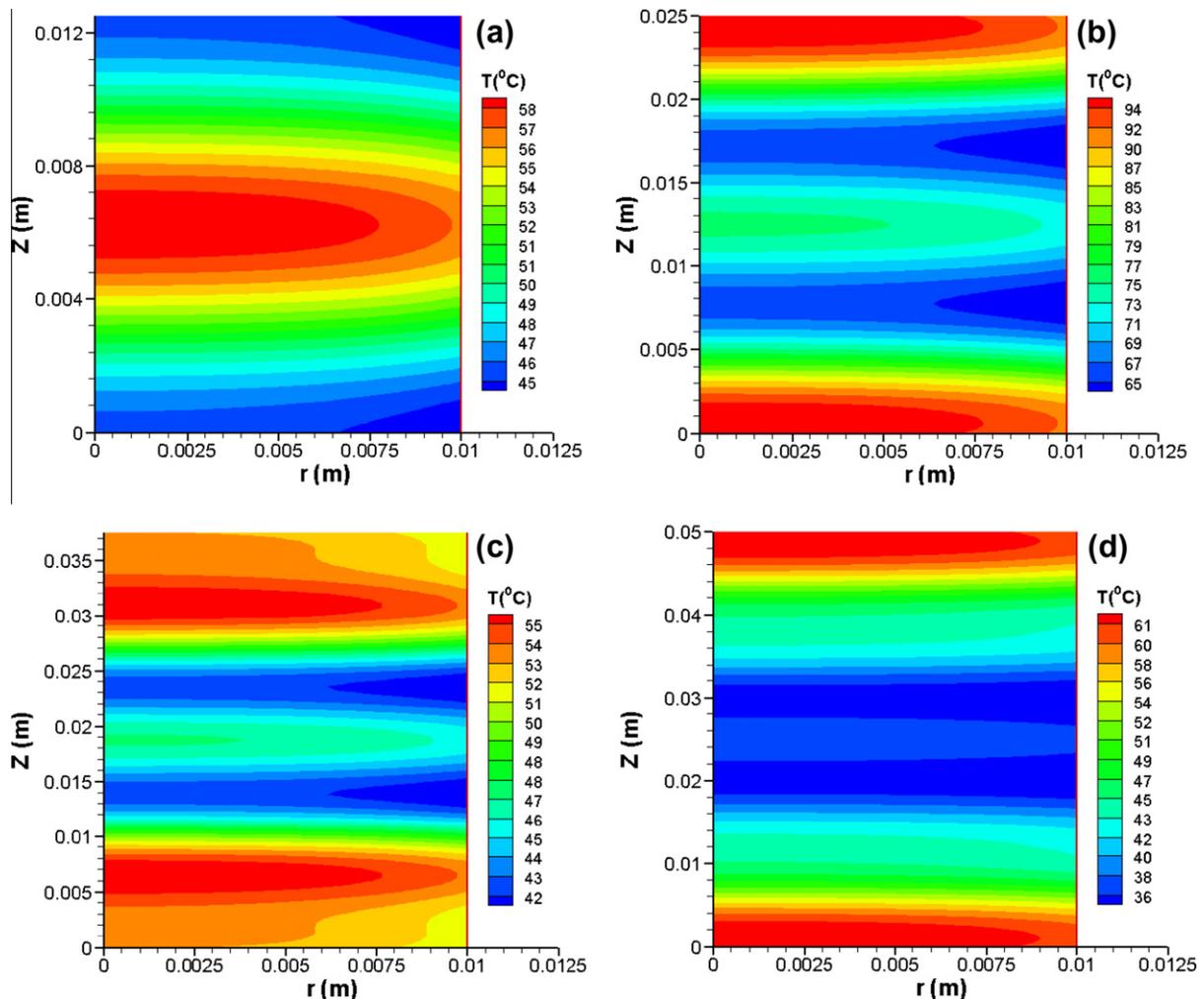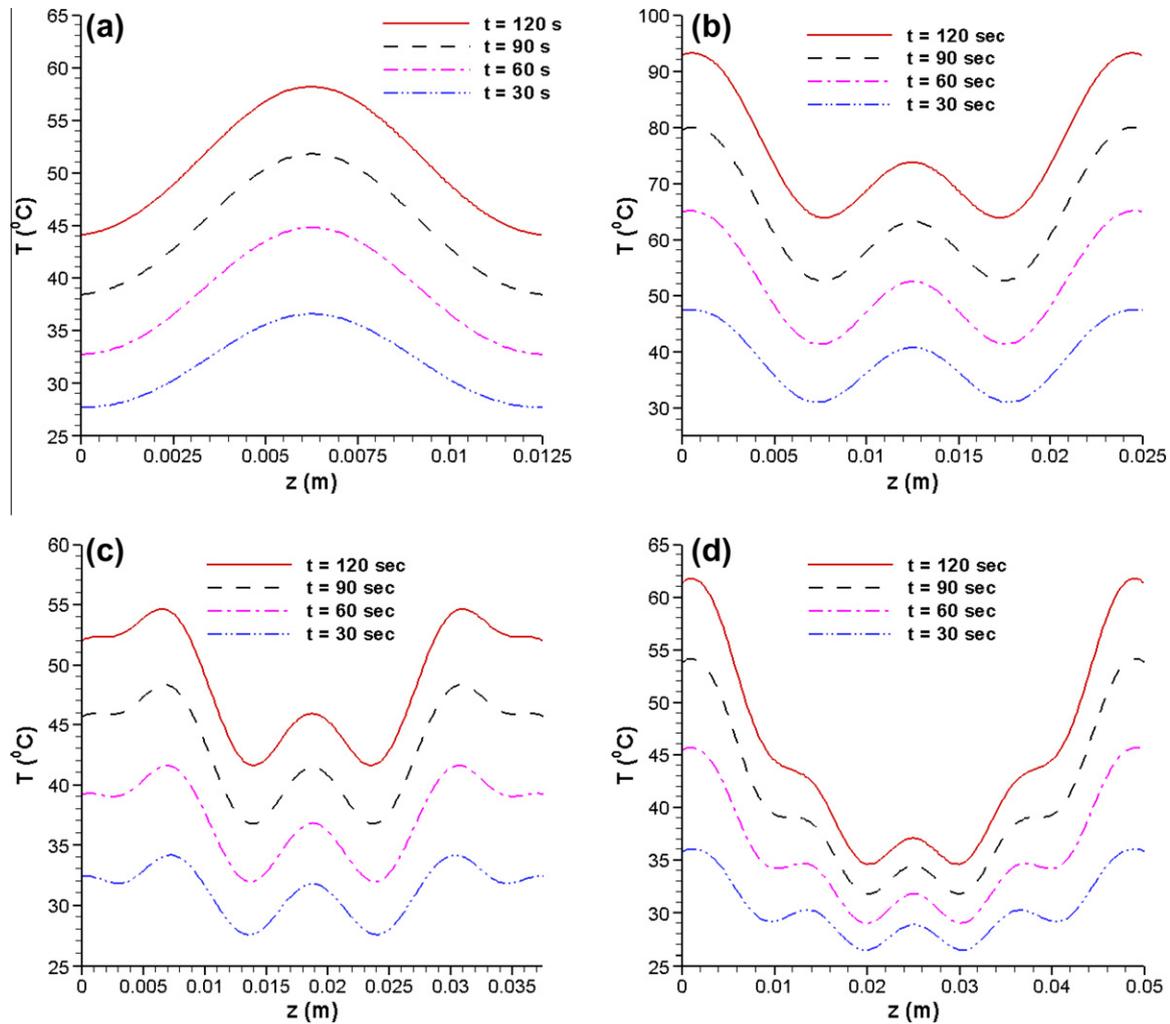


**Fig. 4.** Two dimensional temperature contour for 120 s microwave heating of a 2 cm diameter beef cylinder of length (a) $L$ = 1.25 cm, (b) $L$ = 2.50 cm, (c) $L$ = 3.75 cm, and (d) $L$ = 5.0 cm. Here $h$ = 10 W/m$^2$ K and $f$ = 2450 MHz.

**Fig. 5.** The temperature distribution at different times along the centerline of a beef cylinder. (a) $L = 1.25$ cm, (b) $L = 2.50$ cm, (c) $L = 3.75$ cm, and (d) $L = 5.0$ cm. Here $h = 10$ W/m$^2$ K and $f = 2450$ MHz.

efficiency becomes flat once the number of resonance is six or higher. In that case the temperature is very high at two ends, but almost no temperature increase in the central region of the foodstuff (Fig. 5d). From food safety point of view, this type of heating is unacceptable.

It is interesting to note that the heating efficiencies cannot be used as the only variable in food design for microwave heating. One has to particularly consider the temperature uniformity within the foodstuff in addition to the energy efficiency. The temperature uniformity can be evaluated by thermal range which is the difference between maximum and minimum temperature in the system. Fig. 6a shows that the thermal range is extreme for 2.5 cm long cylinder. At 120 s the thermal range in a food cylinder of length 1.25 cm, 2.50 cm, 3.75 cm and 5.00 cm are 15 °C, 31.25 °C, 13.5 °C and 27.61 °C, respectively. The higher temperature difference indicates uneven temperature within the sample. This particular trend can be explained from the difference in maximum and minimum power absorption within a sample as presented in Fig. 6b. This figure suggest that a more uniform temperature distribution can be obtained for $n < 1$.

We also investigated the effect of heat transfer coefficient on microwave heating for a specific length, diameter, heating time and dielectric properties. The centerline temperature distribution within a 3.75 cm long cylinder is shown in Fig. 7 for different heat

transfer coefficients. This figure indicates that overall temperature as well as the difference between maximum and minimum temperature decrease as the convective heat transfer coefficient increases. At higher heat transfer coefficient the temperature distribution becomes 1D as the heat transfer rate is directly proportional to the heat transfer coefficient. Moreover, the temperature distribution becomes more uniform with increasing heat transfer coefficient, though the heating efficiency decreases significantly. It is interesting to note that the thermal range within the foodstuff decreases as the heat transfer coefficient increases. This means that the heat transfer coefficient of surroundings is also very important for heating uniformity in addition to the thermal properties of the foodstuff. Therefore, the heat transfer coefficient needs to be integrated with other factors such as geometry, dielectric properties to minimize the uneven temperature distribution.

Finally we studied the effect of incident frequency on temperature. The temperature distribution along the centerline of a 2.5 cm long cylinder is shown in Fig. 8 for varying microwave frequency and dielectric properties. The dielectric properties used for this case are shown in Table 1. We specifically choose 2.5 cm long cylinder as it experiences the most non-uniform heating (Fig. 6a) along the axial direction at regular (household microwave oven) microwave frequency of 2450 MHz due to the formation of two resonance peaks (Fig. 2b) in absorbed power distribution.
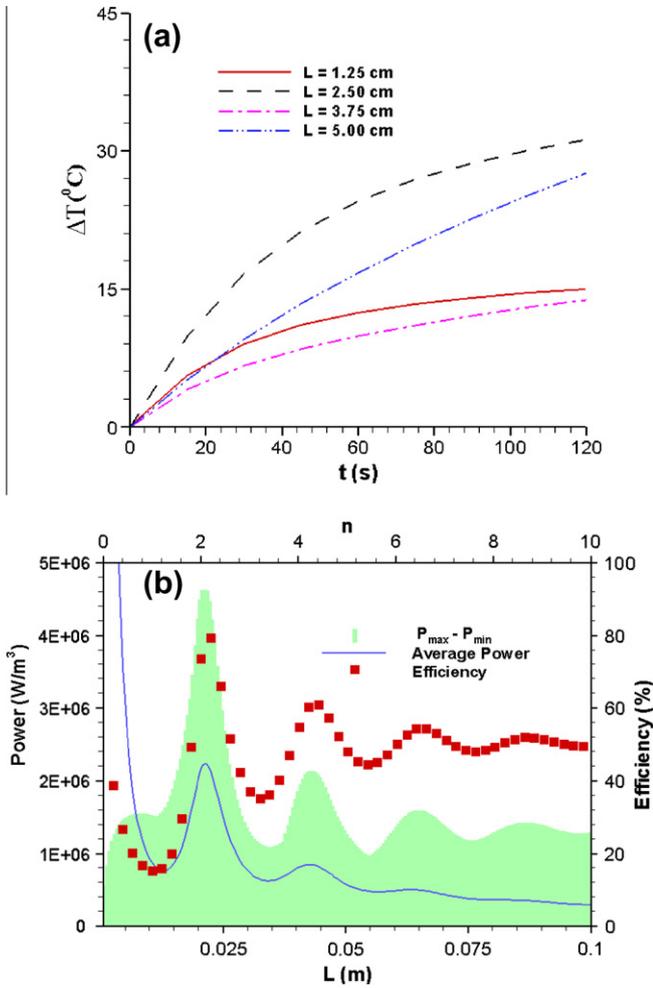
**Fig. 6.** (a) Variation of thermal range with time for different length cylinders. (b) Heating efficiency and power absorption in a beef cylinder as a function of cylinder length. Here $h = 10$ W/m$^2$ K and $f = 2450$ MHz.

However, the number of resonance decreases with increase in wavelength for a particular length. That means at lower frequency (higher wavelength) the number of resonance should be less than 1. As shown in Fig. 8, at frequency of 300 MHz, the beef cylinder provides the most uniform temperature distribution. This figure infers that for specific sample geometry and thermal property, the uniformity in microwave heating can be obtained by controlling the microwave frequency as well as dielectric properties. It is important to note that the degree of non-uniformity in temperature is not linearly related with the number of resonance. Usually, occurrence of two resonances provides the greater non-uniformity in the heating process.

## 5. Summary and conclusions

A closed-form analytic solution is obtained for temperature distribution in food cylinder under microwave heating. The microwave power absorption is computed from Maxwell's equations, and used as a source term in a 3D, unsteady, heat equation. The nonhomogenous heat equation is solved using separation of variables and integral transform techniques. The temperature distributions within the body are presented as a function of cylinder length, radius, heat transfer coefficient, and microwave frequency. This study results in following conclusions.

- The temperature distribution in food cylinders reveals that a slight change in length could cause a significant change in temperature variation within the sample. On the other hand, for a uniform plane wave, the change in the radius has very minor effect on temperature distribution.
- During the microwave heating process, the heat generation rate is much faster than the heat conduction rate.
- The effects of microwave frequency and dielectric constants are very important in obtaining uniform temperature distribution within a sample.
- The core heating or edge heating can be decided by the length of the foodstuff in addition to the dielectric properties and microwave frequency.
- The microwave power absorption and temperature variation within the material are directly related if the heat transfer coefficient is relatively low.
- It is possible to obtain a radially uniform temperature distribution within the foodstuff if the convective heat transfer coefficient is relatively high.
- The heating efficiency is highest when the sample length and microwave frequency allow two resonances in electric field or absorbed power distribution.
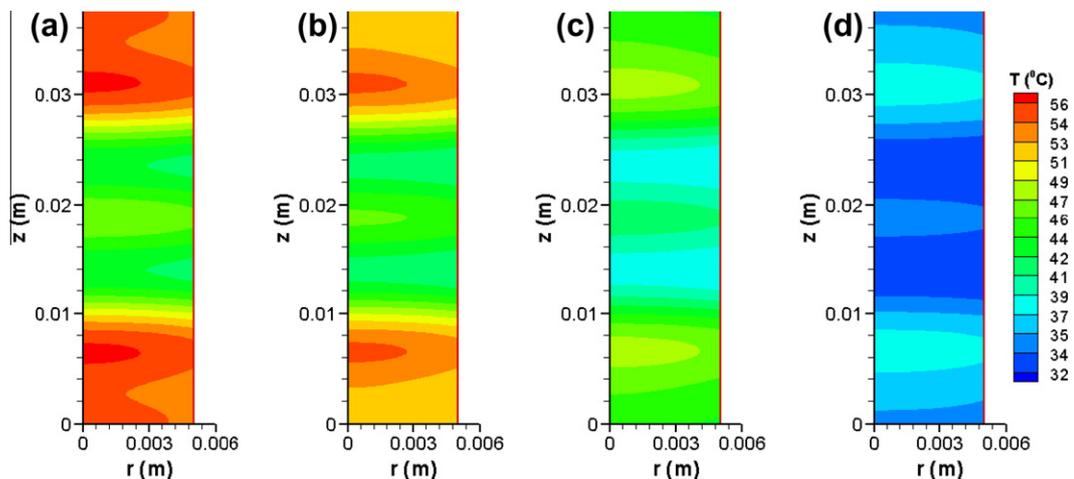


**Fig. 7.** Effect of heat transfer coefficient on temperature contour for 120 s of microwave heating at 2450 MHz frequency. (a) $h = 1.5$ W/m$^2$ K, (b) $h = 10$ W/m$^2$ K, (c) $h = 25$ W/m$^2$ K, and (d) $h = 50$ W/m$^2$ K.
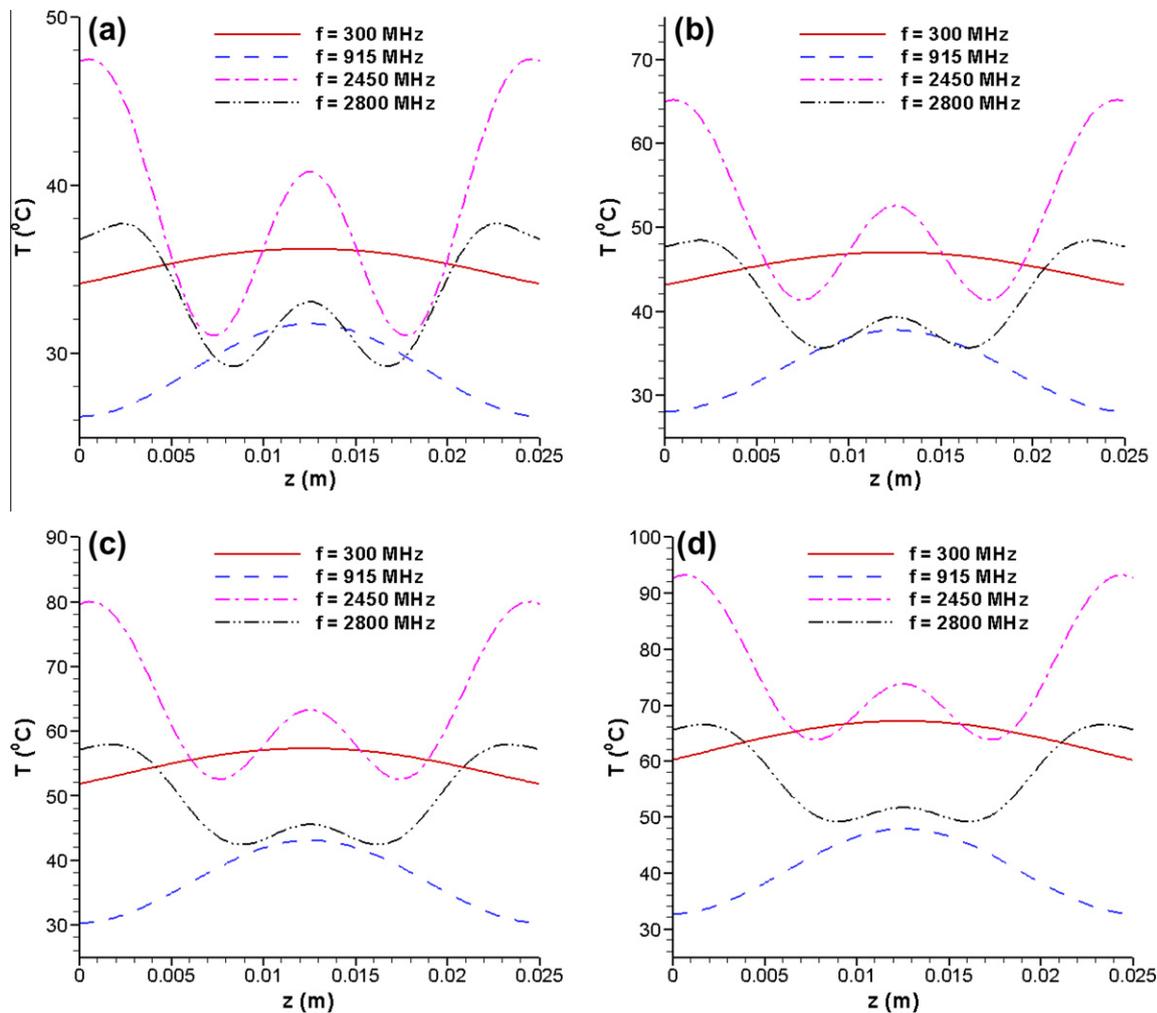
**Fig. 8.** The centerline temperature distribution within a 2.5 cm long beef cylinder for different microwave frequencies. (a) $t = 30$ s, (b) $t = 60$ s, (c) $t = 90$ s, and (d) $t = 120$ s. Here the heat transfer coefficient, $h = 10$ W/m$^2$ K.

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