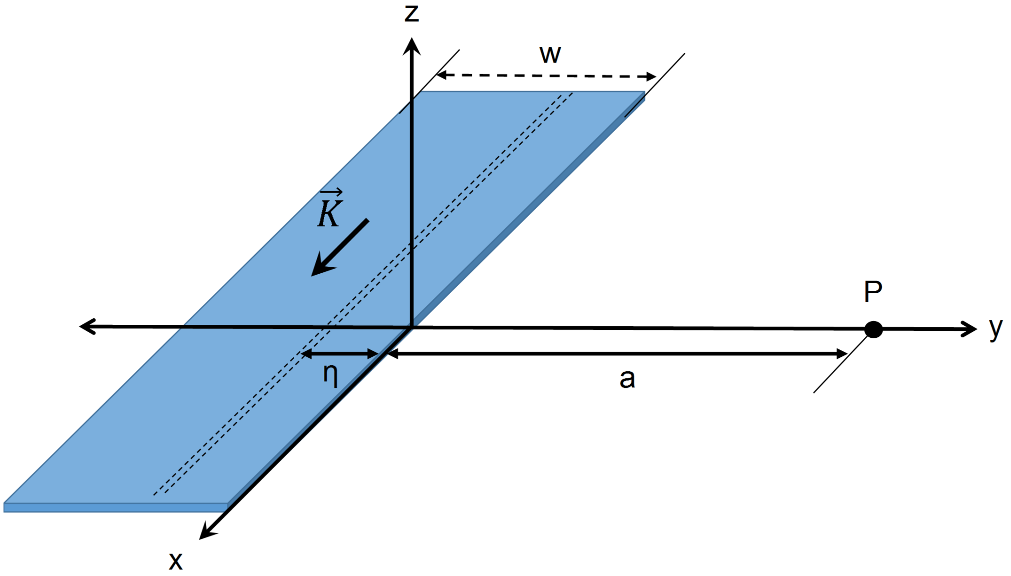
## Magnetic Field due to a Thin Current-Carrying Strip in the Plane of the Strip

**PROBLEM:**
**A thin, infinitely long, conducting strip of width *w* lies in the x-y plane, with its long dimension parallel to the *x*-axis. It carries a surface current *K* in the +*x*-direction.**

**Find the direction and magnitude of the magnetic field at a Point P in the x-y plane a distance *a* from the closest edge of the strip, as shown below.**



(a) Apply Ampere's law to a "wire" defined by a narrow portion of the strip of width $d\eta$ parallel to the x-axis. Assume that the wire intersects the *y*-axis at Point $(0,-\eta)$, where $\eta$ is taken to be positive. Use the right hand rule to determine the direction of the magnetic field at Point P due to this wire. Does the direction of the field depend on the value of $\eta$, assuming that $0 < \eta < w$?

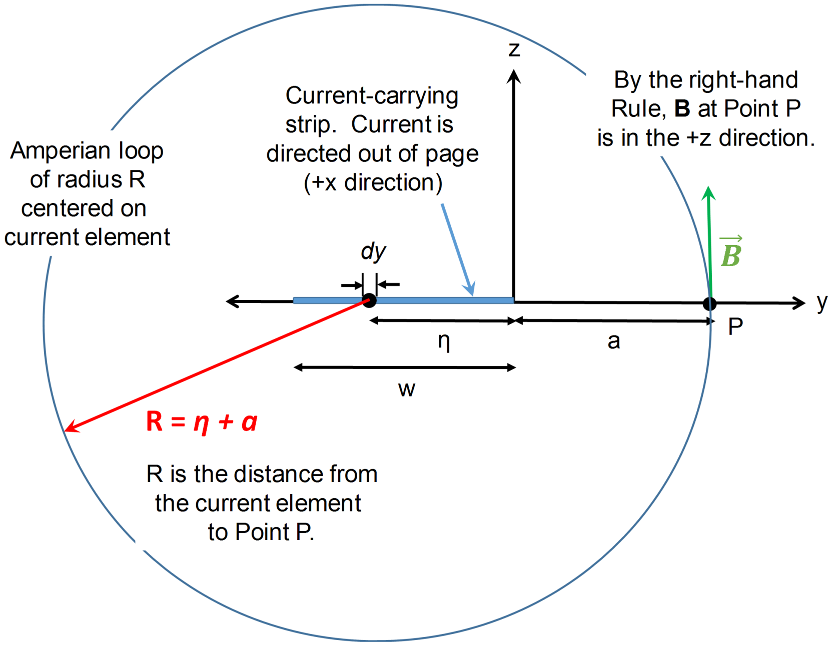
Remember that M uses the symbol I for the square root of –1. Use another symbol for the current. II will work.

**When we applied Ampere's law to an infinitely long, straight wire, we found that:**

$$\int \vec{B} \cdot d\vec{l} = B\, 2\, \pi\, R = \mu o\, II$$

**where, II is the current in the thin wire and R is the distance between the wire and the point where B is measured. In our case, we have a "wire" of width d$\eta$ located a distance $\eta$ from the edge of the strip closest to Point P. The current in this wire is K d$\eta$ (the current per unit**

**width times the width of this wire-like part of the strip). The distance between the wire and Point P is just *a* + *η*. An edge-on view of this configuration is shown below.**



**By the right hand rule, a current flowing in the +*x*-direction will produce a magnetic field in the +*z*-direction at Point P, as long as Point P is located on the positive *y*-axis, as shown.**

**Because the direction of the field at Point P produced by each wire-like element of the strip is the same (+*z*-direction), the magnitude of the field at Point P due to the entire strip will equal the scalar sum of each contribution.**

**Replacing R with (a + *η*) and II with K d*η* in Ampere's law for a thin wire yields**

$$\text{dBatP} = \frac{\mu o\ \text{II}}{2\,\pi\,\text{R}} = \frac{\mu o\ \text{K}\ d\eta}{2\,\pi\,(\text{a} + \eta)}$$

**where dBatP is the contribution of the thin wire to the field at Point P.**

(b) Sum the contributions of the individual wires that make up the strip to determine the magnitude of the magnetic field at Point P. Since M integrates in complex space, you may get a complicated conditional expression. This can be avoided by including assumptions in the Integrate command. For instance, if your function is:

Integrate[Exp[−*η*/a], {*η*,0,w}], you can tell M that *η*, a, and w are positive real numbers by including an assumptions statement:

Integrate[Exp[−*η*/a], {*η*,0,w}, Assumptions->{*η*,a,w} ∈ Reals && {*η*,a,w} > 0].

There are other ways to do this, but putting the assumptions *inside* the Integrate command means that M will forget these assumptions after solving the integral. (The assumptions are “local” to the Integrate command.) This can be handy if you use these variable names later in a different context.

**With $\eta$ as defined above, the wire-like element of the strip closest to P is associated with $\eta = 0$. The wire farthest from P is associated with $\eta = w$. To find the total magnetic field, we integrate from $\eta = 0$ to $\eta = \mathrm{w}$.**

$$\mathbf{BatP} = \frac{\mu o\, K}{2\pi}\int_0^{w}\frac{1}{(a+\eta)}\,d\eta$$

**In the absence of additional information, M treats the variables as complex numbers. One example is shown below.**

In[10]:=

$$\frac{\mu o\ K}{2\ \pi}\int_0^{w}\frac{1}{(a+\eta)}\,d\eta$$

Out[10]=

$$\text{ConditionalExpression}\left[\frac{K\,\mu o\,\left(-\text{Log}[a]+\text{Log}[a+w]\right)}{2\,\pi},\right.$$
$$\left(\left(\text{Re}\left[\frac{a}{w}\right]\geq 0\ \&\&\ \frac{a}{w}\neq 0\right)\ ||\ \frac{a}{w}\notin \mathbb{R}\ ||\ \text{Re}\left[\frac{a}{w}\right]<-1\right)\ \&\&\ \left(\left(\text{Im}[a]\geq 0\ \&\&\ \text{Im}[w]\geq 0\right)\ ||\right.$$
$$\left.\left.\text{Re}[a]\geq\frac{\text{Im}[a]\ \text{Re}[w]}{\text{Im}[w]}\ ||\ \left(\text{Im}[a]\leq 0\ \&\&\ \text{Im}[w]\leq 0\right)\ ||\ \frac{\text{Im}[a]}{\text{Im}[w]}\leq -1\right)\right]$$

**This can be avoided by incorporating the appropriate assumptions, as discussed above. In our case, *a*, *η*, and *w* are all positive real numbers (not imaginary).**

In[11]:=

```
ClearAll["Global`*"]
BatP[a_] =
```

$$\frac{\mu o\ K}{2\ \pi}\ \text{Integrate}\left[\frac{1}{(a+\eta)},\ \{\eta,\ 0,\ w\},\ \text{Assumptions}\rightarrow\{\eta,\ w,\ a\}\in\text{Reals}\ \&\&\ \{\eta,\ w,\ a\}>0\right]$$

Out[12]=

$$\frac{K\,\mu o\ \text{Log}\left[\frac{a+w}{a}\right]}{2\,\pi}$$

(c) Check this result by considering the “narrow strip limit”, *w* → 0, while keeping the total current fixed (K × *w* = II). The result should look like the B field for an infinitely long wire that carries current II. Recall that:

$$\lim_{w\to 0}\left(\log\left(1+\frac{w}{a}\right)\right) = \frac{w}{a}$$

If you like, M will generate the series expansion you need with the Series command. For instance,

Series[Log[1 + *x*],{x,0,3}]

will give the first three terms of the power series expansion of the function Log[1 + *x*] about $x = 0$. It uses the traditional O[*x*]$^4$ notation to indicate that the order of magnitude of next (fourth) term grows as $x^4$. If you don't want this addition (which M won't evaluate), you can write:

Normal[Series[Log[1 + *x*],{*x*,0,3}]]

**First, perform the expansion.**

In[13]:=
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Series[Log[1 + x], {x, 0, 3}]
```

Out[13]=
$$x - \frac{x^2}{2} + \frac{x^3}{3} + O[x]^4$$

In[14]:=
$$\texttt{Normal[Series[Log[1 + x], \{x, 0, 3\}]] /. x} \rightarrow \frac{w}{a}$$

Out[14]=
$$\frac{w}{a} - \frac{w^2}{2\,a^2} + \frac{w^3}{3\,a^3}$$

**As w gets smaller and smaller, the first term will eventually dominate.**
**Then**

$$\textbf{BatP} = \lim_{w \rightarrow \text{really small}} \left(\frac{K\,\mu o}{2\,\pi}\,\mathrm{Log}\left[\frac{a+w}{a}\right]\right) = \frac{K\,\mu o}{2\,\pi}\,\frac{w}{a}$$

**Since *K w* = II, where II is the total current in the narrow strip,**

$$\textbf{BatP[a]} = \frac{\mu o\,\mathrm{II}}{2\,\pi\,a}$$

**which is just the magnitude of the B-field produced at Point P by an infinitely long wire carrying a current II— a successful check.**

(d) JFL == Just for Laughs: Determine BatP if you let w -> ∞. (use M's Limit function)

In[15]:=
$$\texttt{BatP[a\_]} = \frac{K\,\mu o\,\mathrm{Log}\left[\frac{a+w}{a}\right]}{2\,\pi};$$
```
Limit[BatP[a], w → ∞]
```

Out[16]=
$$K\,\mu o\,\infty$$

**As w becomes >> a, Log[$\frac{a+w}{a}$] goes as Log[w], which goes to infinity as w → ∞.**

**So, although the "wires" making up the additional parts of the sheet (as you increase w) are further and further away from the point P (so the contribution of these parts to B is decreasing), additional wires keeps adding more and more to B.**

**Below is a plot of BatP for a finite wmax; it looks like it might not go to infinity but....wrong. It does. you can change wmax if you wish and see what it looks like (it just keeps going up!).**

In[17]:=
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a = 1; wmax = 100;
Plot[Log[(a + w)/a], {w, .01, wmax}]
```

Out[18]=