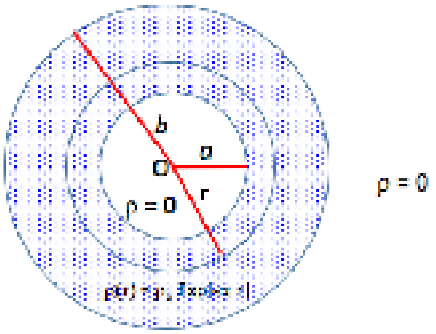
E-Field and Electric Potential for a thick shell of charge with a $\rho[r] = \rho_o \, \mathrm{Exp}[-\alpha\, r]$

A thick, non-conducting spherical shell of inner radius *a* and outer radius *b* has a volume charge density

$\rho[r] = \rho_o \, \mathrm{Exp}[-\alpha\, r]$ for a ≤ r ≤ b; assume ρ_o and α are positive.
$\rho[r] = 0$ elsewhere (for r < a and r > b).
Find $\vec{E}$ and V at all r.



(a)Just for reference, make a plot of ρ[r] (we need to assume some values for the constants; I'll use a = 1; b = 2; α = 0.5; ρo = 1)

```
ClearAll["`*"]
a = 1; b = 2; α = 0.5; ρo = 1; (*  just to get you started *)
```

We want to Find $\vec{E}$. Let's not beat around the bushes - we will use Gauss's Law to find $\vec{E}$.

(b) But first, in the text cell below, show that the vector $\vec{E}$ is radial in direction *and* that the magnitude of $\vec{E}$ depends only on the distance r from the origin O, where r = $|\vec{r}|$, the magnitude of $\vec{r}$. Remember r is a distance, therefore a scalar. This allows us to write: $\vec{E}[\,\vec{r}\,] = E[\,r\,]\,\hat{r}$. (Note carefully where the symbols have vector vs. scalar status.) Motivation: This greatly simplifies applying Gauss's Law to find $\vec{E}$.

```
this is a text cell – type in your response  You might want to include a fig-
ure.
```

First, verify the radial direction of $\vec{E}$.

Show that the magnitude of $\vec{E}$ at the point P being a function of r the scalar r — the length of $\vec{r}$) only.

For obvious reasons, ONCE we go through this lengthy exercise (or similar arguments for other symmetries) we generally and happily say "because of symmetry, . . . " and quickly move on.

(c) State why we make these arguments.

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(d) Find $\vec{E}[\vec{r}]$ for r < a (Probably will not need to use M). A diagram would be useful.

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(e) Find $\vec{E}$inside_shell[$\vec{r}$], i.e., for a ≤ r ≤ b (Suggest you set up both sides of Gauss's Law, then use M.)

`this is a text cell – type in your analysis of the LHS and RHS of Gauss's Law here:`

HINTS: I wrote out both sides of Gauss's Law and then used Solve to find the function Eshell[r] (the magnitude of E inside the shell of charge)

The Solve statement looked like this: sol = Solve[LHS ==RHS, Eshell[r]].

The vector field $\vec{E}$inside_shell[$\vec{r}$] is simply Eshell[r] $\hat{r}$ (the magnitude times the unit vector).

```
(* Input Cell *)
ClearAll["`*"]
ρ[rprime_] = ρo Exp[-α rprime]; (* just a starter  *)
```

(f) Now find $\vec{E}$ outside the shell, $\vec{E}$outside_shell[$\vec{r}$], that is, $\vec{E}$ for r > b.

`this is a text cell – type in your reasoning here`

More Hints: Again, HINTS: I wrote out both sides of Gauss's Law and then used Solve to find the function Eoutsideshell[r] (the magnitude of E outside the shell of charge)

The Solve statement looked like this: sol = Solve[LHS ==RHS, Eoutsideshell[r]].

The vector field $\vec{E}$outside_shell[$\vec{r}$] is simply Eoutsideshell[r] $\hat{r}$ (the magnitude times the unit vector).

```
(* Input Cell *)
```

(g) Use If functions to define the magnitude of $\vec{E}$[r],(which I write as EE[r]), for *all* r (An alternative approach is to use PieceWise[ ].)

(ok, as a gift — this will work: `EE[r_] = If[r < a,0,If[a≤r≤b,Eshell[r],If[r > b,Eoutsideshell[r]]]]`)

Chose values for the constants and Plot EE[r] from r = 0 to some r > b.

(I used the following values: `a = 1; b = 2; α = 0.5; ρo = 1; ϵo = 1;`)

```
(* Input Cell *)
```

Just for comparison, below is a plot of the Charge Enclosed (Qenclosed[r]) by a spherical Gaussian Surface of radius r for our particular ρ[r]

To beat a very dead horse, for a spherical symmetric ρ[r]: $\mathrm{EE}[r] *(4\,\pi\,r^2) = \dfrac{\mathrm{Qenclosed}[r]}{\epsilon o}$

```
Clear[a , b , α , ρo , ϵo]
Qenclosed[r_] = If[r < a, 0, If[a ≤ r ≤ b,
     ∫_a^r ρ[rprime] 4 π rprime^2 ⅆrprime, If[ r > b, ∫_a^b ρ[rprime] 4 π rprime^2 ⅆrprime]]];
a = 1; b = 3; α = 0.5; ρo = 1; ϵo = 1;
plotQ = Plot[Qenclosed[r], {r, 0, 6}, AxesLabel → {"r", "Qenclosed[r]"}];
lines2 =
  Graphics[{Line[{{a, 0}, {a, 35}}], Line[{{b, 0}, {b, 35}}], Text["r = a", {a, 7},
     Background → LightRed], Text["r = b", {b, 7}, Background → LightGreen]}];
Show[plotQ, lines2]
```

The relation between this curve and the above plot of E[r] is: $\mathrm{E}[r] = \dfrac{\mathrm{Qenclosed}[r]}{4\,\pi\,\epsilon o\;\; r^2} = k\,\dfrac{\mathrm{Qenclosed}[r]}{r^2}$

Note that the constant value of Qenclosed for r ≥ b equals the total charge contained in the whole shell which is:

$$\mathrm{Qtotal} = \int_a^b \rho[\mathrm{rprime}]\; 4\,\pi\;\mathrm{rprime}^2\; d\,\mathrm{rprime}$$

(h) Produce a VectorPlot of EE[r]·$\hat{r}$ in two dimensions (i.e., in a plane passing through the polar axis of the sphere). Remember, you will need to explicitly define 2D vector functions (for the three regions: r < a; a ≤ r ≤ b, and r > b) and convert them to Cartesian coordinates to use M’s VectorPlot function.

```
(* Input Cell *)
Clear[a , b , α , ρo , ϵo]
(* this simply clears the values assigned to these constants  *)
```

(i) Interpret the resulting 2D Vector Plot for the given spherically symmetric charge distribution:

$$\rho[r] = \begin{cases} 0 & \text{for } r < a \\ \rho o\ \mathrm{Exp}[-\alpha\, r] & \text{for } a \le r \le b \\ 0 & \text{for } r > b \end{cases}$$

`(this is a text cell - type in your interpretation)`

(j) Now derive the electric potential V[r] for all regions. It is very easy to show that if $\vec{E}$[r] = Er[r] $\hat{r}$, V will be a function of r only. We will go with that.

`(this is a text cell - type in your interpretation)`

First, it might be helpful if we copy and paste clean spherical coordinate versions of Er[r] (the magnitude of $\vec{E}$[r] = Er[r] $\hat{r}$) for the three regions:

$$\text{Eoutsideshell[r]} = \frac{\left(e^{-a\,\alpha}\,(2+a\,\alpha\,(2+a\,\alpha))+e^{-b\,\alpha}\,(-2-b\,\alpha\,(2+b\,\alpha))\right)\rho o}{r^2\,\alpha^3\,\epsilon o} \qquad \text{for } r > b$$

$$\text{Er[r]} = \qquad \text{Eshell[r]} = \frac{(e^{-a\,\alpha}\,(2+a\,\alpha\,(2+a\,\alpha))+e^{-r\,\alpha}\,(-2-r\,\alpha\,(2+r\,\alpha)))\,\rho o}{r^2\,\alpha^3\,\epsilon o} \qquad \text{for } a \le r \le b$$

$$\text{Ecavity[r]} = \qquad 0 \qquad \text{for } r < a$$

FIRST do r > b:

```
(*  Input cell *)
Clear[a, α, b, ϵo, ρo]
                         1
Eoutsideshell[r_] =  ─────────  (ⅇ^(-a α) (2 + a α (2 + a α)) + ⅇ^(-b α) (-2 - b α (2 + b α))) ρo;
                      r² α³ ϵo
(* a reminder *)
$Assumptions = (b ≤  rprime ≤ ∞) && rprime ∈ Reals && r ∈ Reals && r ≥ b;
(* useful for helping M integrate *)
```

SECOND, inside the shell (a ≤ r ≤ b):

To find Vshell[r], we integrate Eoutside[r] from ∞ to b (which equals Voutsideshell[b]) and then integrate Eshell[r] from b to r (to get what I call Vinsideonly[r]). (Of course we insert the – sign.)

Vshell[r] is the *sum* of the two contributions. The limits on these integrals are consistent with moving the test charge from ∞ to an r inside the shell, a ≤ r ≤ b. Note: I have changed the assumptions to aid the Vinsideonly[r] integration. I recommend using Simplify near or at end.

Third, inside the empty cavity (r < a): Since there is no charge enclosed, the electric field in the cavity is zero and the extra work to move the test charge from r = a to anywhere in the empty

cavity will also be zero. Thus, the potential inside the cavity Vcavity[r] = Vshell[a] (a constant!).

Here are the calculations to obtain Vshell[r]

```
(*  Input cell  *)
```

$$\text{Eshell[r\_]} = \frac{1}{r^2\,\alpha^3\,\epsilon o}\left(e^{-a\,\alpha}\,(2 + a\,\alpha\,(2 + a\,\alpha)) + e^{-r\,\alpha}\,(-2 - r\,\alpha\,(2 + r\,\alpha))\right)\rho o;$$

```
$Assumptions = (a ≤  rprime ≤ b) && rprime ∈ Reals &&
   (a ≤  r ≤ b) && r ∈ Reals && r > 0 && α ∈ Reals && α > 0;
```

THIRD, inside the empty cavity (r < a):

```
(*  Input cell --enter expression for Vcavity[r_] *)
```

(k) Now plot V[r] for all three regions combined.

```
(*  Input cell *)
```

(l) Plot the magnitude of $\vec{E}$[r], V[r], the charge density ρ[r], and the Qenclosed[r] together in a single graph. You may want to adjust some of the amplitudes so they stay usefully scaled.

Plotting the magnitude of E[r], V[r], the charge density ρ[r], and the Qenclosed[r] over all the regions to compare their behavior (amplitudes have been adjusted to simplify comparison of curve behaviors):

```
(*  Input Cell *)
ρ[r_] = If[r < a, 0, If [a ≤  r ≤ b, ρo Exp[-α r], If[r > b, 0]]];
(* just some help *)
```

(m) Make some interpretive remarks about these plots (worry not about repeating comments from above).

```
(this is a text cell - type in your interpretation)
```

(n) Generate a 2D Contour Map of the potential over all three regions. {Hint: ContourPlot uses Cartesian Coordinates—you can use your general V[r] above and simply replace it with

V[r] /. r → $(x^2 + y^2)^{\frac{1}{2}}$}

Using Show, superimpose this ContourPlot with the VectorPlot of EE[r]·$\hat{r}$ above.

```
(*  Input Cell *)
```

(o) Interpret your combined plot.

(this is a text cell - type in your interpretation)