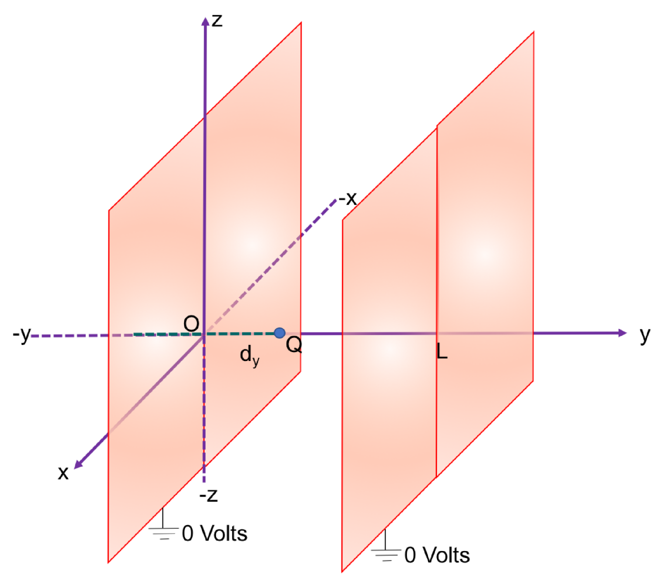
Point Charge Between Two Parallel Conducting Plates

PROBLEM:

A point charge Q is located between two infinite grounded parallel conducting sheets at a distance of d_y from the plate passing through the origin, as shown (note the coordinate system used including the position of the origin O).

The separation between the plates = L. We wish to find the electrostatic potential V[x,y,z] for points between the plates.

Single Point Charge Q positioned between two infinite, grounded, parallel conducting sheets



(a) What classic technique or method that is used for determining V and E for charges near certain symmetric geometries of conductors is your likely choice??

<Enter your answer in this text cell>

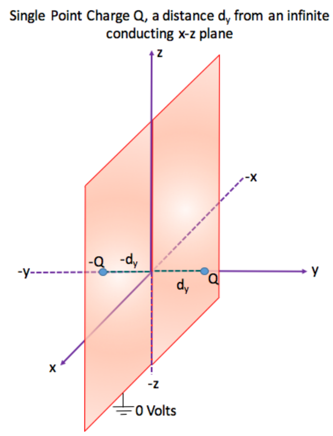
(b) In this type of problem, what role do boundary conditions play in finding V and E?

<Enter your answer in this text cell>

So above, *Your Response* included *Images charges??* Brilliant! The challenge is to find a set of point charges that will produce a zero-potential on both conducting planes. Hints:

We use the results of using a single image charge for the case of a single real charge Q a distance $+d_y$ in front of an infinite grounded conducting (x-z) plane. The correct image charge in this example, $Q_{image} = -Q$, is placed a distance d_y *behind* the plane. This will yield a zero potential along the entire x-z plane located at y = 0.

Single Plane; point charge:



Reminder: To find V[x,y,z] for y > 0 ONLY, we take simply add $V_{(Q)} + V_{(-Q)}$;

$$\text{i.e.,} \quad V[x,y,z] = k\,Q\left(\frac{1}{\sqrt{x^2+(y-d_y)^2+z^2}} + \frac{1}{\sqrt{x^2+(y+d_y)^2+z^2}}\right)$$

(c) Determine the magnitudes and positions of the image charges such that both planes are at zero volts potential [Superhint: image charges will all be located along the ± y axis.]

<Enter your answer in this text cell>

(d) Stick your results into M; use ± nmax for the limits on the sums over n instead of ± ∞.

```
(* Input code below *)
Evaluation → QuitKernal → Local; (*  this does a super
 scrub of constants and variables that might be problematic  *)
ClearAll["Global`*"] (* Leave the ClearAll statement *)
```

(e) Put in values for the constants. I used: k = 1; Q = 1; L = 2; dy = 0.3; nmax = 20.
Make a 3D Plot of V in the z = 0 plane (i.e., of V[x,y,0]).

```
(* Input code below *)
```

This is the V sought in the region where it is defined (between the plates) for the L, dy (0.3), and Q chosen. Using a finite nmax is, of course, an approximation. However, varying nmax to any value > ~ 8, it is very hard to see any visual change in this plot.

Now let's show the potential for both the REAL region between the plates and in the region where you're not supposed to look or go! (y < 0 and y > L). Then you see the regions where there are image charges.

Although not obvious, all of the image charges are contributing to V between the plates. I've outlined the "real region" with a black rectangle.

```
(* Execute code below *)
Print[Style["This is a Plot of the Potential due to Q and a few of the
    image charges (again, staying in the z = 0 plane).   Only the
    outlined region (for 0 > y > L) between the plates is 'real'.
    Use your mouse to rotate and/or resize. ", Brown, Bold, 16]]
plotty = Plot3D[V[x, y, 0], {x, -2 L, 2 L}, {y, -4 L, 4 L}, PlotRange → {-5, 5},
   BoxRatios → {8, 16, 10}, MaxRecursion → 6,
   PlotLabel → Style[Framed["V[x,y,0] showing image charges"], 16, Blue, Bold,
     Background → Lighter[Orange]], AxesLabel → {"x", "y", "V[x,y,z]"}];
liny = Graphics3D[{Thick, Line[{{-2 L, 0, 0}, {2 L, 0, 0},
      {2 L, L, 0}, {-2 L, L, 0}, {-2 L, 0, 0}}]}];
Show[
 plotty,
 liny]
```

To repeat, in this plot, only the region outlined in black is in the space between the plates where V is actually defined/valid. Also, it is obvious that there are both + and – Q image charges corresponding to the + and - peaks in V. For d_y such that Q is close to either plate, the system looks like a string of dipoles separated by a distance 2L.

(f) Before pursuing further interpretation of the V plot, I want to use Manipulate and show the above for different values of d_y (moving the charge from one plate towards the other). Run this cell and vary d_y by using the slider.

Note: When you change d_y, it may take some time to recompute V and update the plot. I also want to plot V of the point charge by itself, Vpoint[x,y,0], for comparison.

M Note: I used an option BoxRatios→Automatic which scales the axes (most importantly (x,y)) 1:1. This insures that the plot is not distorted.

Changing dy on one plot changes the other simultaneously so you can easily compare.

Use your mouse to rotate and modify the size of the plots for best viewing.

```
(* Execute code below *)
Clear[dy];
```

$$\text{Vpoint}[x\_, y\_, z\_] = \frac{+k\,Q}{\sqrt{\mathbb{R}\text{positive}[0].\mathbb{R}\text{positive}[0]}};$$

```
Print[
 Style["This is a Plot of the desired Potential V[x,y,0] BETWEEN THE PLATES as
    a function of x,y in the z = 0 plane.  With the
    slider, we can Manipulate the distance (dy) of the
    point charge from the y = 0 plate.  ", Red, Bold, 16]]
Print[Style["      \nSUGGESTION: Rather than using the slider,
    IT'S BEST TO JUST ENTER A NEW dy IN THE WINDOW displaying
    the magnitude of dy and hit RETURN. ", Blue, Bold, 14]]
Print[Style["      (When you change dy, it will take some time
    to recompute V and update the plot)", Green, Bold, 14]]
Manipulate[dy; Plot3D[V[x, y, 0], {x, -2 L, 2 L}, {y, 0, L},
  PlotRange → {0, 5}, BoxRatios → Automatic, MaxRecursion → 6,
  PlotLabel → Style[Framed["V[x,y,0] Between Plates"], 16, Blue, Bold,
    Background → Lighter[Orange]], AxesLabel → {"x", "y", "V[x,y,0]"}],
 {dy, .1, 1.9, Appearance → "Open"}, LocalizeVariables → False]

Print[Style[
  "Below is a Plot of the Potential of the Single POINT CHARGE Q only BETWEEN
    THE PLATES (no image charges) as a function of x,y in the z = 0 plane
    where we can Manipulate the distance (dy) of the point charge from
    the y = 0 plate.  NOTE:  Changing dy on either of these Manipulate
    plots CHANGES BOTH -- this is an M thingy. ", Red, Bold, 16]]


Manipulate[dy; Plot3D[Vpoint[x, y, 0], {x, -2 L, 2 L},
  {y, 0, L}, PlotRange → {0, 5}, BoxRatios → Automatic, MaxRecursion → 6,
  PlotLabel → Style[Framed["Vpoint[x,y,0] Between Plates"], 16, Blue, Bold,
    Background → Lighter[Orange]], AxesLabel → {"x", "y", "V[x,y,0]"}],
 {dy, .1, 1.9, Appearance → "Open"}, LocalizeVariables → False]
```

(g) Now interpret these results (e.g., where are the plates? Shape of plots; compare?? Putting charge in center vs. very close to plates? Enter in the text cell below:

<Enter interpretation in this text cell>

The plates are located along the planes associated with y = 0 and y = L. Remember that the values of the scalar FUNCTIONS V[x, y, 0] and Vpoint[x, y, 0] are what is being plotted in the vertical direction. ALSO, remember that we have set z = 0 in these functions. We are exploring the V's between the plates in the same plane that hosts the real point charge Q.

In both plots, the infinity at the position of the real point charges {0, dy, 0} is clearly observed. when dy = L/2, both plots show the expected symmetry (you could call it the lack of ϕ dependence). Vpoint maintains that symmetry for all dy because it's simply the V of a single point charge.

V[x,y,0] for dy very small shows the ESSENTIAL fall of V to zero all along the grounded plate at y = 0 (and at the plate located at y = L). The point charge alone has no idea the grounded plate exists so exhibits the 1/r fall off as expected. Clearly, Vpoint does not fall to zero at the plates.

Same for dy very close to L near the other plate.

IF you wanted to explore V and Vpoint for other values of z (above or below the x-y plane), say zo, you can change the V[x, y, 0] and Vpoint[x, y, 0] to V[x, y, zo] and Vpoint[x, y, zo]. One result will be that the infinities in the V's will vanish (i.e., V[x, dy, zo] and Vpoint[x, dy, zo] are finite. Why? Because for zo ≠ 0, $\sqrt{\mathbb{R}\text{positive}[n] . \mathbb{R}\text{positive}[n]}$ and $\sqrt{\mathbb{R}\text{negative}[n] . \mathbb{R}\text{negative}[n]}$ do not go to zero. Thus, their reciprocals are finite.)

CONTOURS OF V:

A "vanilla" contour plot of V in the z = 0 plane looks like this:

```
(* Execute code below *)
dy = 0.3;
contourplot = ContourPlot[V[x, y, 0], {x, -2 L, 2 L}, {y, 0, L},
  AspectRatio → Automatic, Frame → True, FrameLabel → {"x axis", "y axis"}]
```

M blindly determines the contours it will display; to display more contours we have to play around. I have kluged a set of contours (values of V) using a bunch of Tables Joined together ("cons" is a list of these values). Once executed, you can point your mouse at any contour on the plot and read its value.

```
(* Execute code below *)
cons = Join[Table[20 - i, {i, 0, 20}], Table[1 - 0.1 i, {i, 0, 9}],
  Table[0.1 - 0.01 i, {i, 0, 9}], Table[0.01 - 0.001 i, {i, 0, 9}]]

Print[
 Style["Here is a ContourPlot of the Potential BETWEEN THE PLATES as a function
    of x,y IN THE z = 0 plane (which contains Q):", Purple, Bold, 16]]
contoury = contourplot = ContourPlot[V[x, y, 0], {x, -2 L, 2 L}, {y, 0, L},
   PlotRange → {0, 10}, ColorFunction → Hue, Contours → cons, Frame → True,
   FrameLabel → {"x ", "y "}, PlotLabel → "V[x,y,0] Contours",
   MaxRecursion → 2, AspectRatio → Automatic, ImageSize → 500]
```

(h) Now find E (suggest you use EE) in this segment. (Hint: use –Grad in Cartesian Coordinates). Remember you have to have nnmax defined; I keep nnmax = 30.

```
(* Input code below *)
Clear[k , Q , L, dy]; (* Leave the Clear command *)

nmax = 30;
EE[x_, y_, z_] = - Grad[V[x, y, z], {x, y, z}];
(* if you really want to see it, remove the ; *)
```

(i) Now we want to do some plotting of the field, comparing with the above ContourPlot, and a look at the induced charges on the plates. Let me walk you through this.

First, plotting the vector E field is challenging (to see all of it's features). First, let's show a StreamPlot to get a general feel for the field. Here, I put it on top of the ContourPlot above (I've stayed with dy = 0.3). This code refers to EE[x_,y_.z_], which you hopefully have defined in Part (i).

```
(* Execute code below *)
k = 1; Q = 1; L = 2; dy = 0.3;
EEE[x_, y_] = { EE[x, y, 0][[1]], EE[x, y, 0][[2]]};
streamy = StreamPlot[EEE[x, y], {x, -2 L, 2 L}, {y, 0, L},
   AspectRatio → Automatic, ImageSize → 500, StreamColorFunction → "Rainbow"];
Show[contoury, streamy]
```

The Streamlines are curves that trace tangents to the E field which (symbolically) indicate the flow of the field (think fluids — i.e., the flow velocity field). Magnitudes of E are not shown.

The plot above shows that the directions of E are normal to the contour lines, consistent with the directions of maximum gradient (E = – Grad V). Also note that the streamlines are approaching each plate (at y = 0 and y = L) in a normal direction. The plates are both equipotentials (0 V) and so E should be normal to the plates. Note also that the directions of E is consistent with negative charge on both plates (induced by the presence of +Q); i.e., the field lines originate on + Q and terminate on the negative charges distributed on the two plates.

Somewhat misleading is the missing evidence of E Field directly in the path between +Q (at {0, dy} and {0,0}. The contours are changing in large steps of V and are closely spaced (decreasing as y -> 0). E should be BIG.

Again, it is challenging to show a VectorPlot of V between the plates. Here is the Vanilla Version (it shows One Big Vector!):

```
(* Execute code below *)
VectorPlot[EEE[x, y], {x, -2 L, 2 L}, {y, 0, L},
 AspectRatio → Automatic, Frame → True, FrameLabel → {"x ", "y "}]
```

Not much better, but with some tricks I can milk out a little more of the field:

```
(* Execute code below *)
EEEE[x_, y_] = If[Norm[EEE[x, y]] > 50, {0, 0}, EEE[x, y]];
vecy = VectorPlot[EEEE[x, y], {x, -2 L, 2 L}, {y, 0, L},
  AspectRatio → Automatic, ImageSize → 500, Frame → True, FrameLabel → {"x ", "y"}]
```

The bottom line, is the field is indeed the largest in the line between the location of the +Q and the closest point on the bottom plate {0,0} and falls off rapidly in all other directions.

Let's simply plot the amplitude of the (x and y) components of he E field in the z = 0 plane along the path from the origin {0,0} [in the center of the x = 0 plate] to the point {0, L} [in the center of the y = L plate]; the arrow shows the y position of the +Q.

```
(* Execute code below *)
dy = 0.3;
dypos = Graphics[{Arrow[{{dy, 1600}, {dy, 100}}], Text["dy", {dy, 1800}]}];
plotEamplitudes =
  Plot[{EE[0, y, 0][[1]], EE[0, y, 0][[2]]}, {y, 0, L}, PlotRange → {-2000, 2000},
   AxesLabel → {"y", "EEx and EEy"} , PlotLegends → {"EEx", "EEy"}];
Show[plotEamplitudes, dypos ]
```

You can see that EE_x is zero (the straight blue line along the y axis) for all y along this path (due to symmetry); same for EE_z, by the way. The switch in sign seen in EE_y is expected because of the switch in direction of EE at y = d_y where, of course, infinities on each side

are observed.

Now look at the induced charge distributions on the two electrode surfaces. You recall that at a conducting surface, $\sigma = \frac{E_{\perp}^{\text{surface}}}{\epsilon o}$. The sign of σ is + if $E_{\perp}^{\text{surface}}$ points out of the surface and - if $E_{\perp}^{\text{surface}}$points towards the surface.

It's not hard to see that $E_{\perp}^{\text{surface}}$ at the y = 0 plane is EE[0,y][[2]] and $E_{\perp}^{\text{surface}}$ at the y = L plane is EE[0,L][[2]]. (The [[2]] selects the y component of EE)

So we can quickly generate the charge densities on the two planar electrodes in the problem (I call them sigmazero = "sigma_y = 0_plane" and sigmaL = "sigma_y = L_plane"). At both surfaces, σ is negative. I had to put in a – sign for sigmaL to force sigmaL to be negative. [This reflects that fact that the surface normal for the surface at y = 0 points to the right (into the region of interest), while the surface normal for the surface at y = 0 points to the left (also into the region of interest).]

```
(* Execute code below *)
sigmazero = EE[x, 0, z][[2]] / ϵo;
sigmaL  = - EE[x, L, z][[2]] / ϵo;
 ϵo = 1;

Plot3D[sigmazero, {x, - 2 L, 2 L}, {z, -2 L, 2 L},
 PlotRange → {0, -20}, AxesLabel → {"x", "z", "sigma_y = 0_plane"}]
Plot3D[sigmaL, {x, - 2 L, 2 L}, {z, -2 L, 2 L}, PlotRange → {0, -1},
 AxesLabel → {"x", "z", "sigma_y = L_plane"}]
```

Note that the sigma scales are different — with Q close to the y = 0 plane, the sigma on the y = L plane is considerably smaller in line with the much reduced E field at y = L relative to y = 0. For L = 2 and dy = 1 (Q right in the middle), here are the charge distributions. Use your mouse to tilt and rotate the plots.

```
(* Execute code below *)
dy = 1;
sigmazero = EE[x, 0, z][[2]] / ϵo ;
sigmaL  = - EE[x, L, z][[2]] / ϵo ;
 ϵo = 1;

Plot3D[sigmazero, {x, - 2 L, 2 L}, {z, -2 L, 2 L},
 PlotRange → {0, -2}, AxesLabel → {"x", "z", "sigma_y = 0_plane"}]
Plot3D[sigmaL, {x, - 2 L, 2 L}, {z, -2 L, 2 L}, PlotRange → {0, -2},
 AxesLabel → {"x", "z", "sigma_y = L_plane"}]
```

As expected, they are the same due to the symmetric positioning of Q between the plates.

To rap it up, Q induces negative surface charge distributions (sigmazero and sigmaL) on the two plates. The field between the plates is determined by the sum of the fields due to these two charge distributions and the original +Q.

IF by magic we had known sigmazero and sigmaL [as functions of x,y,z] and the position of +Q, we could, in principle, calculate EE between the plates. Certainly numerical calculations would be possible using the tools of Griffiths Chapter 2.

BUT we did not know sigmazero and sigmaL. Instead, we were able to use the Image Charge Method to calculate a Solution to Laplace's Equation (without solving it!) that satisfies the BCs. By *Uniqueness* we then have derived the desired V[x,y,z]. From this V, we calculated E[x,y,z] and the surface charge distributions on the plates.

Done.