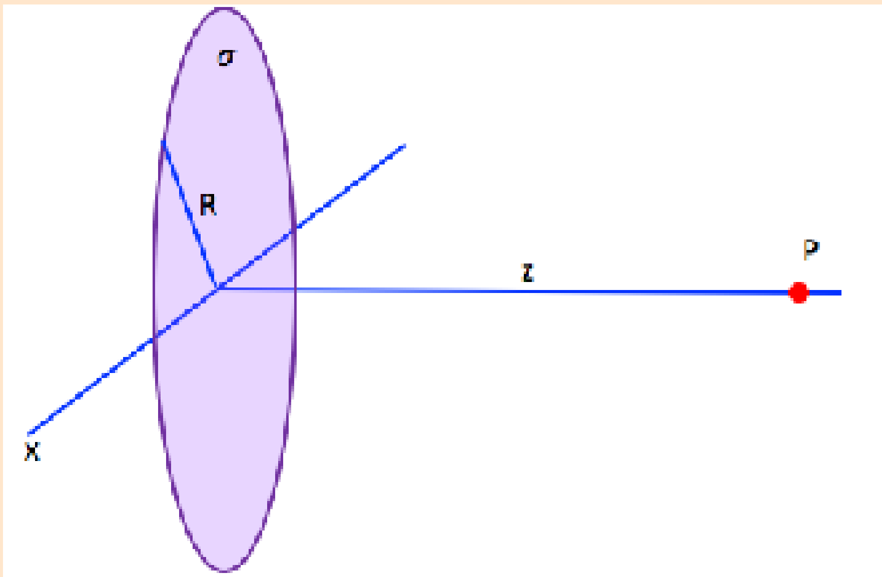
Solution to Potential and E-Field of Charged Disk ALONG Symmetry Axis

Problem: Consider a disk of radius R with a uniform charge density σ. Find the Electric Field due to this charge distribution on the axis of symmetry (z axis) for both z > 0 and z < 0. Denote the distance along the z axis from the center of the disk (O) to the point P (on the z axis) by z.



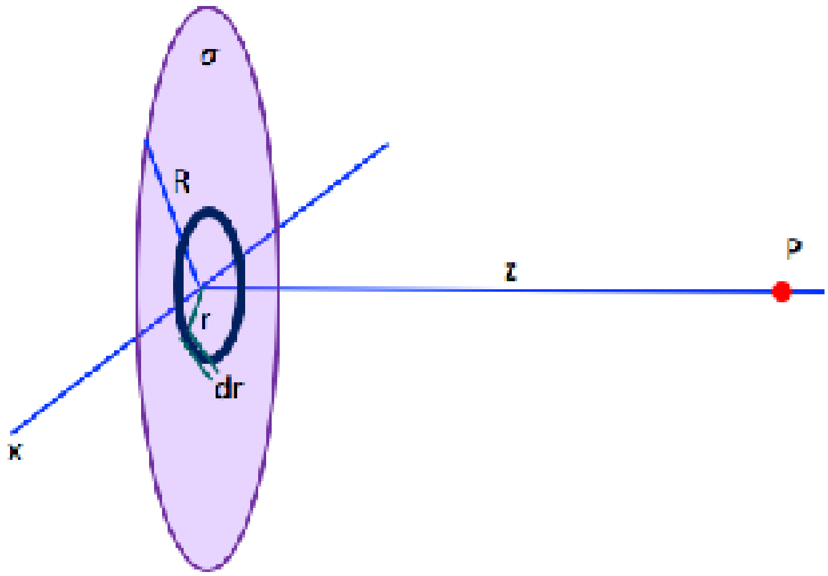
A couple of reminders:

1. (* This is a comment *)

and

2. If you get some weird results at some point, try going back and re-executing all the previous cells to reset your functions and variables.

(a) Start by finding a differential of electric potential dVring[z] (dVring as a function of z) at the point P due to a **ring of charge** of a radius between r and r + dr. Then integrate over r to find Vdisk[z].



First, the potential of a ring of charge with a uniform linear charge density and radius R was found to be equal to:

$V_{ring} = \frac{2 R k \pi \lambda}{\sqrt{R^2+z^2}}$; $\left(k = \frac{1}{4 \pi \epsilon 0}\right)$

But the equivalent line charge density is $\lambda = \frac{Q_{ring}}{2 \pi R}$

which $\Rightarrow V_{ring} = \frac{2 R k \pi \frac{Q_{ring}}{2 \pi R}}{\sqrt{R^2+z^2}} = \frac{k Q_{ring}}{\sqrt{R^2+z^2}}$.

For our ring: Q_{ring} = $Q_{\text{ring of radius r and thickness dr}}$ = $2 \pi r\, dr\, \sigma$; (you could call this dQring, a differential of charge, if you wished)

***ding-dong*: note that here we use r, not R because we are going to integrate over r.**

V_{ring} **turns into a differential,** dV_{disk}**:**

The differential of the potential due to this ring of radius r = $dV_{disk}[z]$ = $\frac{k\, 2 \pi r\, dr\, \sigma}{\sqrt{r^2+z^2}}$.

We integrate over r from 0 to R to get Vdisk[z].

(b) Enter your equation for dVring[z] in an appropriate integral written in Mathematica and thereby find the equation for Vdisk[z]. A series of assumptions (more than you need) are provided that make the integration run like a champ. They start with: $Assumptions = ...

Reminder: *To execute each cell, click your mouse anywhere inside the cell and then hit Shift-Return (Shift-Enter)*

Below is an input cell you can use for finding V[z] :

```
(* Input Cell *)
ClearAll["`*"]
$Assumptions = R > 0 && R ∈ Reals &&
   z ∈ Reals && z ≠ 0 && k ∈ Reals && k > 0 && σ ∈ Reals && σ > 0;
    (*  ENTER in this cell (below this comment) your dV_disk[z]
 into the appropriate integral  --reminder:  Limits needed *)
```

You should now have the scalar function Vdisk[z] defined as an algebraic expression which contains the parameters k, R, λ, and the variable z.

(c) Assume some values for k, R, and σ; plot Vdisk[z] from some $-z_o$ to $+z_o$.

I chose k =1, R=1, and σ=1 and I set z_o = 10. [I suggest you use a σ >0 to help interpret your result.]

```
(* Input Cell -- enter your code and evaluate*)
```

(d) Clear[k,σ, R] and then evaluate Vdisk[z] to see that it looks ok.
(I'll do it for you):

```
Clear[k, σ, R]
Vdisk[z]
```

(e) Interpret the plot of Vdisk[z] you obtained.

Enter your Discussion (this is a text cell):

(FYI- extra info)

You should note that in both directions, the potential falls to very small values for z >> R; the disk starts looking like a point charge. The total charge on the disk is:

$$Q_{\text{disk}} = (\text{Area of Disk}) * \sigma = \pi R^2 \sigma$$

Here we plot V_{disk}[z] and the potential of a point charge: $\frac{k\,Q_{\text{disk}}}{\text{Abs}[z]}$ (We insert the Abs[z] to replace z because for $+Q_{\text{disk}}$, V of a point charge is positive for both +z and -z; Abs[z] takes care of this).

```
Qdisk = π R^2 σ;
k = 1; σ = 1; R = 1;
Plot[{Vdisk[z], k Qdisk/Abs[z]}, {z, -20, 20}, PlotRange → {0, 10}]
```

As expected, at sufficiently large |z|, the two potentials (the disk and the point charge) are indistinguishable; both go to zero for z -> ∞.

(f) So let's handshake on the presence of a term in Vdisk[z] containing Abs[z]. To find the E field from the potential M or we must take derivatives. M does not like taking the derivative of Abs[z] with respect to z.

We make life a lot easier for M and for ourselves if we divide the solution into two parts for z>0 and z<0. We can combine them with an If statement. It will look like this:

Vdisk[z_] = If[z < 0, (*Stick in here your Vdisk[z] for z < 0*), If[z > 0, (*Stick in here your Vdisk[z] for z > 0*)]].

M is happy taking derivatives of Vdisk[z] in the form of the If statement; it simply performs it for each part separately.

Thinking carefully, determine the appropriate Vdisk[z] for the two signs of z and enter into the input cell below your resulting Vdisk[z] [in the form of an If statement]. Execute the cell so M has Vdisk[z] defined. Stick in a Clear[k, σ, R] to keep things honest.

```
(*  Input Cell  - write your Vdisk[z] for both signs of z;
you can use an If statement OR the Piecewise function  *)
Clear[k, σ, R]
```

(g) For comparison with the plot of Vdisk[z] above, Plot this new Vdisk[z] for the same k, σ, R and for the same z_o, over the range $-z_o$ to $+z_o$.

As before, I used k =1, σ=1, R = 1; I set z_o = 10.

```
(*  Input Cell *)
```

The plots for the two forms of Vdisk[z] are the same, i.e., for both ± z; the SIGN of z automatically handled this for the first version.

(h) Now find the Electric Field, E_{disk}. I used M's Grad function in Cartesian Coordinates (which generates a VECTOR).

```
Clear[k, σ, R];
Vdisk[z] (* just to check to see it is still defined *)
```

This generates a 3D vector (for both signs of z $_o$ where the z component is the only non-zero component (reasonable: Vdisk[z] depends only on z; therefore, at the point P, $\overrightarrow{\text{Edisk}}$[z] = Ediskz[z] $\hat{z}$

Using Vdisk[z] for the potential we get an If statement in the resulting $\overrightarrow{\text{Edisk}}$[z]. Looking carefully, we can conclude that $\overrightarrow{\text{Edisk}}$[z] for z < 0 equals (-) $\overrightarrow{\text{Edisk}}$[z] for z > 0. [Hopefully you agree that: $\overrightarrow{\text{Edisk}}$[z] = Ediskz[z] $\hat{z}$.]

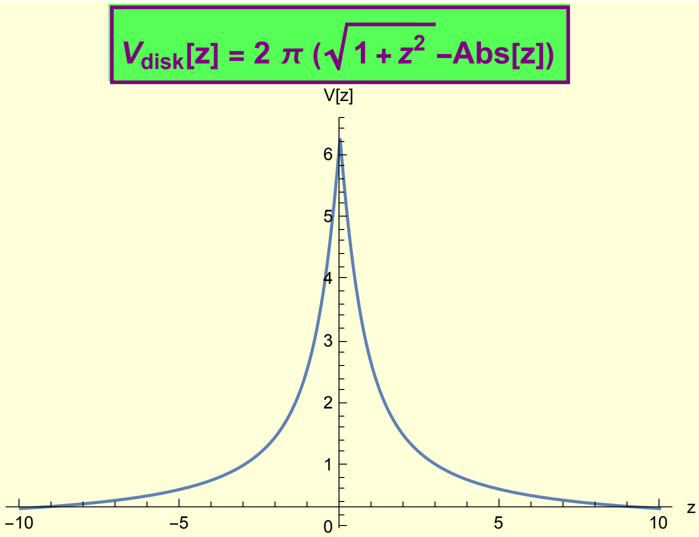
(i) Now Plot Ediskz[z] for the same k, σ, R and for the same z $_o$, i.e., over the same range -z $_o$ to + z $_o$ you used above. [You will need to grab the z component of $\overrightarrow{\text{Edisk}}$[z], which we call Ediskz[z].]

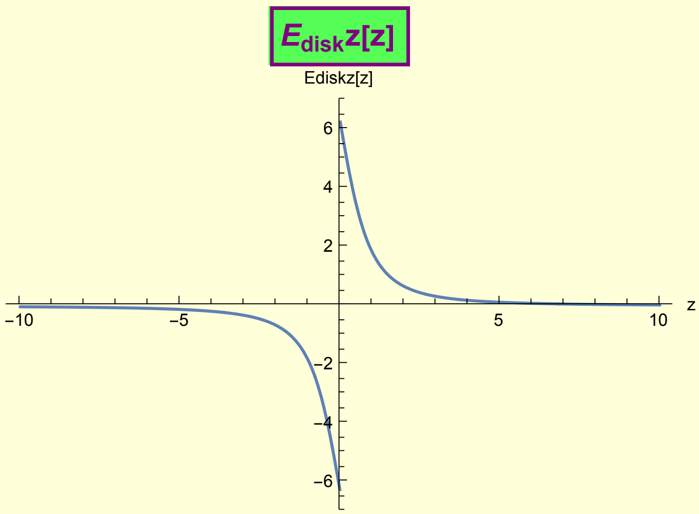
```
(*  Input Cell *)
Ediskz[z_] =  Edisk[z] [[3]]
(*  quick way to pull out the z component of Edisk[z] *)
k = 1; σ = 1; R = 1;  (* need these defined for the Plot  *)

Plot[Ediskz[z], {z, -10, 10}, PlotRange → {-7, 7},
 AxesLabel → {"z", "Ediskz[z]"}, PlotLabel →
  Style[Framed["E_disk z[z]"], 16, Purple, Bold, Background → Lighter[Green]]]
```

(j) Write down a brief interpretation/discussion about the two plots (e.g., SIGNS and the sign of λ). How are these two plots are related (hint: SLOPE of one of them)??

Enter your Discussion (this is a text cell):

Interpretation: First, we compare this curve with the above Vdisk[*z*], **(I've copied and pasted these plots from above to help you get started:)**





Your Turn:

(k) No Brainer - Click inside the cell below (or select it by clicking on the bracket to the right) and execute it (Shift-Return); Answer Boxes will appear; Click on the one you think is correct answer for this question:

Question: Off this symmetry (z) axis, I expect V_{disk} and $\vec{E}_{disk}$ to depend on z only. [Live it up! Click both.]

Imagine moving a +q test charge around the disk with uniform + σ at various x,y,z values off the z axis. I think you can see that the off axis solution: $V_{disk}[x, y, z]$ **depends in general on x,y, AND z.**

```
Button[
 "1  I agree.  Off the symmetry axis, Vdisk and Edisk depend on z only",{Print[
   "  Wrong --The symmetry of the problem is broken: in Cartesian Coordinates,
     we therefore expect x and/or y dependence to creep in. "]}]
Button["2 I disagree; Off the symmetry axis, Vdisk and EEdisk
   generally do not depend on z only ",
 {Print["  Correct -- The symmetry of the problem is broken; in
     Cartesian Coordinates, we therefore expect x and/or y
     dependence to creep in.\n\nIn Spherical Coordinates one
     would expect θ dependence in V and E, but no ϕ dependence."]}]
```

So No. 2 is the correct answer.