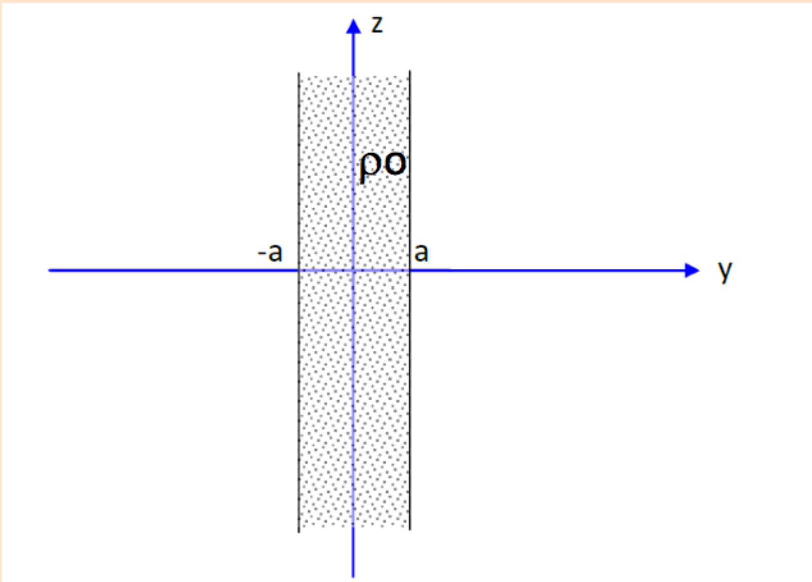
Electric Field and Potential for an infinite slab with uniform charge density ρ

PROBLEM: Consider an infinite slab parallel to the x-z plane (normal to the y axis) of uniform charge density ρ and thickness 2*a*, as shown:



Find the Electric Field and Potential everywhere.

(a) First, present your arguments for treating this problem as one dimensional.

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The infinite slab can be thought of a set of parallel infinite sheets of uniform surface charge density σ (= ρ dy where dy is the 'thickness of the charge sheet). The field for such a charge sheet is independent of x and z and normal to the charge sheet, therefore normal to the x-z plane (for σ>0, E points away from the plane of charge).

Superposition will lead to an E Field that will also be independent of x & z and normal to the x-z plane. Thus, a one D problem (y dependent).

We wish to derive the E field, $\vec{E}$(y), at all values of y.

(b) Start with E outside the slab.

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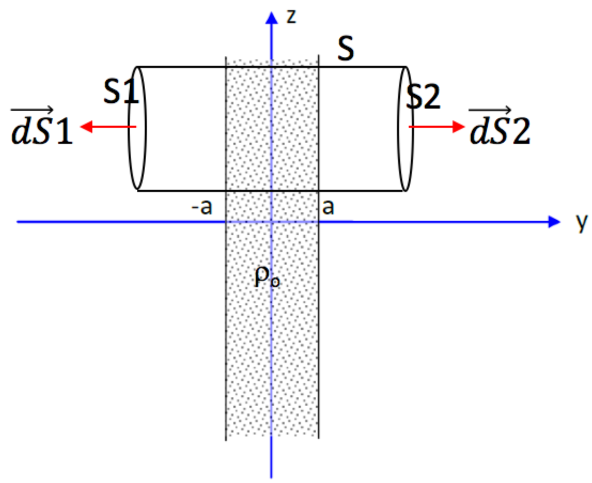
Suggest using Gauss's law.

First we find the field OUTSIDE using a pillbox wider than 2a and both ends of area A placed OUTSIDE

of the charge slab.



$\vec{E}_{\text{outside}}$ will be in the ±y direction and therefore normal to the ends S1 and S2 (i.e., in the $\pm\hat{y}$ direction). Let A be the area of both S1 and S2.

Gauss's Law, $\int_S \vec{E}.d\vec{S} = \int_{S1} \vec{E1}.d\vec{S}1 + \int_{S2} \vec{E2}.d\vec{S}2 = \frac{1}{\epsilon 0}$ **(Charge Contained inside *S*)**

The charge contained in the pillbox = ρ A (2a).

All the E's are || to the dS's; For a given y, the E's are constant.

$E_{\text{outside}}\,A + E_{\text{outside}}\,A = \frac{1}{\epsilon_o}\,\rho\,A\,(2a)$ **or**

On the Right: $\vec{E}_{\text{outpos}} = \frac{1}{\epsilon_o}\,\rho\,a\,\hat{y}$

and

On the Left: $\vec{E}_{\text{outneg}} = -\frac{1}{\epsilon_o}\,\rho\,a\,\hat{y}$

(c) Now find E inside the slab.

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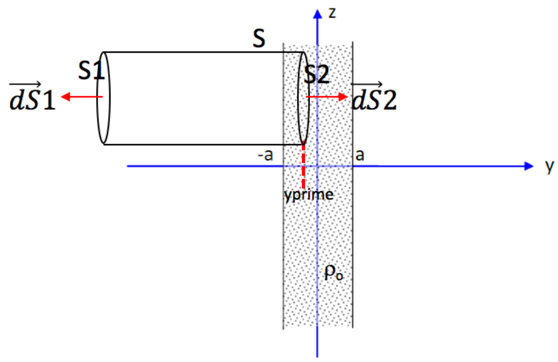
Now inside the slab (–a ≤ y ≤ a). We redraw S to have one end of the pillbox inside the charged region located at y = yprime. I label the end of the pillbox inside S2 (on the right). $\vec{E1}$ at S1 = $\vec{E}_{\text{outneg}} = -\frac{1}{\epsilon_o}$ ρ a $\hat{y}$ on the left.



Gauss's Law, $\int_S \vec{E}.d\vec{S} = \int_{S1} \vec{E1}.d\vec{S1} + \int_{S2} \vec{E2}.d\vec{S2} = \frac{1}{\epsilon 0}$ **(Charge Contained inside** S**)**

E1 A + $E_{inside}[y]$ **A =** $\frac{1}{\epsilon 0}$ **(Charge Contained inside** S**)**

$\frac{1}{\epsilon_o} \rho$ **a A +** $E_{inside}[y]$ **A =** $\frac{1}{\epsilon 0} (\rho (a + y) A)$ **Note: SIGN of y critical here in determining** $Q_{enclosed}$

$E_{inside}[y] = \frac{1}{\epsilon 0} (\rho (a + y) A) - \frac{1}{\epsilon_o} \rho a A$

$\vec{E}_{inside}[y] = \frac{\rho y}{\epsilon_o} \hat{y}$ **Note: the sign of y is critical in determining the direction of** $\vec{E}_{inside}[y]$

(d) Plot the magnitude of EE[y] for both inside and outside the slab.

```
(* Input Cell - Enter your code here -- start with a ClearAll["`*"]    *)
ClearAll["`*"]
Einside[y_] = rho y / epsilonzero; (*  NOTE x takes care of the SIGN)   *)
Eoutneg[y_] = - rho a / epsilonzero;
Eoutpos[y_] = rho a / epsilonzero;
rho = 1; epsilonzero = 1; a = 2;
EE[y_] = If[y < -a, Eoutneg[y], If[-a ≤ y ≤ a, Einside[y], If[y > a, Eoutpos[y]]]];
EEplot = Plot[EE[y], {y, -2 a , 2 a}, AxesLabel → {"y", "EE(y component)[y]"},
   PlotLabel → Style[Framed["y component of the E-Field vs. y"],
     16, Red, Background → Lighter[Yellow]]];
lineleft = Graphics[{Thick, Dashed, Blue, Line[{{-a, -2 a - .5}, {-a, 2 a + .5}}]}];
lineright = Graphics[{Thick, Dashed, Blue, Line[{{a, -2 a - .5}, {a, 2 a + .5}}]}];
Show[EEplot, lineleft , lineright]
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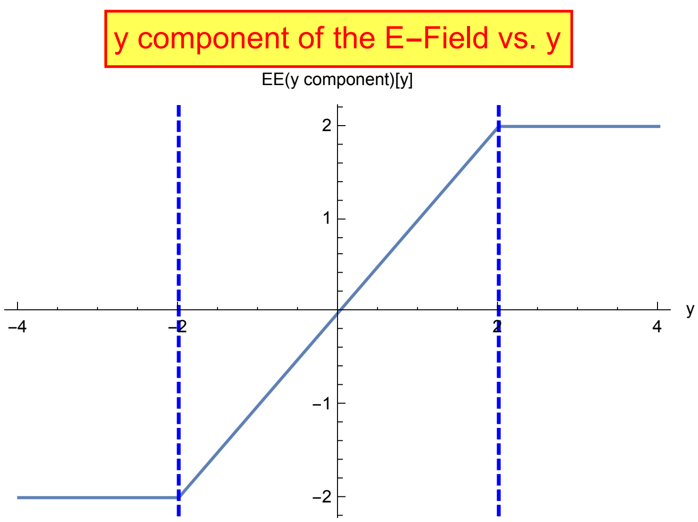

(e) Comment on this plot

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First, the blue vertical lines are the edges of the slab of charge. Note to the left of the origin, E[y] is < 0, meaning that the vector E[y] points to the left. To the right of the origin, E[y] is > 0, meaning that the vector $\vec{E}$[y] points to the right. We assumed for this plot that rho = ρ = +1 (positive charge). So outside the slab $\vec{E}$ should point away from the slab which it does. INSIDE: think of the slab as a bunch of identical, parallel sheets of positive charge. If you are at a point to the left of the origin (say y = -1), you would have more sheets of + charge on the right than on the left. Therefore the superposition of the fields from these sheets would lead to a field pointing left. The linear behavior of $|\vec{E}|$ is reasonable in terms of the Number of sheets on the left vs. on the right.

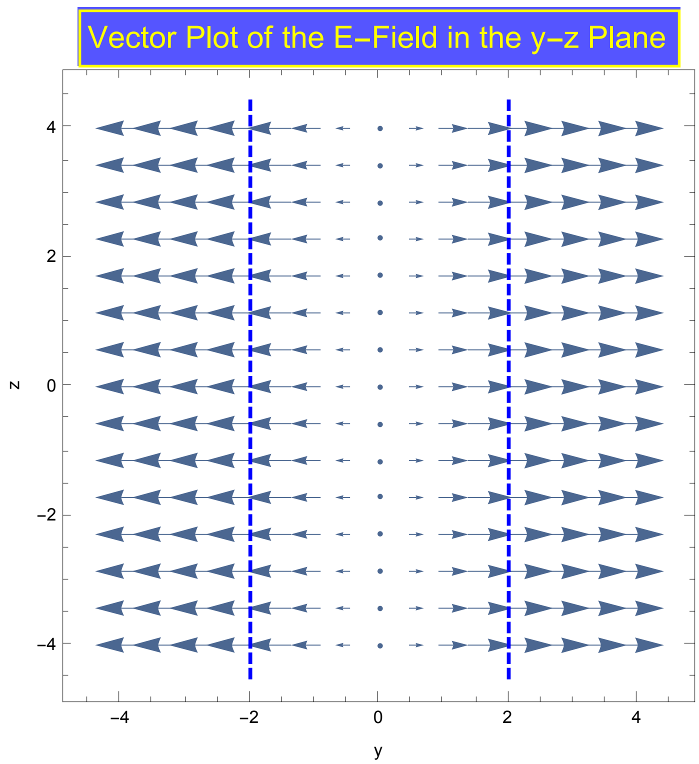
Not that E[y] is continuous at the edges of the slab. There is a boundary condition on $\vec{E}$ at an interface (let's use "left" and "right" for position, assuming an interface, such as ours, parallel to the x-z plane).

$$E_{\text{normal-left}} - E_{\text{normal-right}} = \frac{\sigma}{\epsilon o}$$

For a continuous slab of charge, the surface is considered free of charge (because ρ changes abruptly from 0 to ρ with y) so we take $\sigma = 0$. Thus, $E^{\text{interface}}_{\text{normal-left}} = E^{\text{interface}}_{\text{normal-right}}$, which is what is observed in the graph at both interfaces (at y = ± a).

(f) Now generate a 2D Vector Plot of the E-Field both inside and outside of the slab of charge (assume a positive ρ).

```
(* Input Cell - Enter your code here *)
vecplot =
  VectorPlot[{EE[y], 0}, {y, -2 a , 2 a}, {z, -2 a, 2 a}, FrameLabel → {"y", "z"},
   PlotLabel → Style[Framed["Vector Plot of the E-Field in the y-z Plane"],
     16, Yellow, Background → Lighter[Blue]]];
Show[vecplot, lineleft, lineright]
```



(g) Comment on this vector plot

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Again, the blue vertical lines are the edges of the slab of charge. Again, we assumed for this plot that rho = ρ = +1 (positive charge). We see that the direction of the E vectors are consistent with the amount of +

charge on the left side of a given point (y value) chosen vs. the amount of + charge on the right side of that point. E = 0 at y = 0 because the two amounts of charge balance. [Of course, we cannot forget that the slab extends to ∞ in both the x and z directions].

So E grows in magnitude (linearly with y) from zero at y = 0 to a maximum at y = ± a. Outside, both left and right, the fields are of constant magnitude (uniform). It probably need not be mentioned, but such uniform fields require that the charge be of infinite extent in {x, z}.

Although not so obvious in the vector plot, $E_{\text{normal-left}}^{\text{interface}} = E_{\text{normal-right}}^{\text{interface}}$ at both y = ± a.

(h) Derive and plot the 1D Electric Potential vs. y for the above E Field.

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Starting again with the E field expressions, we assume that the potential at y = 0 is zero; Potin[0] = 0.

WHY? For |Eout| constant vs. y, we would conclude that E at infinity = this constant. I.e., it would take an infinite amount of work to move a unit test charge from infinity to the slab. Thus, we could not set Potin[∞] = 0. We therefore adopt y = 0 as a reference point where we set Potin[0] = 0.

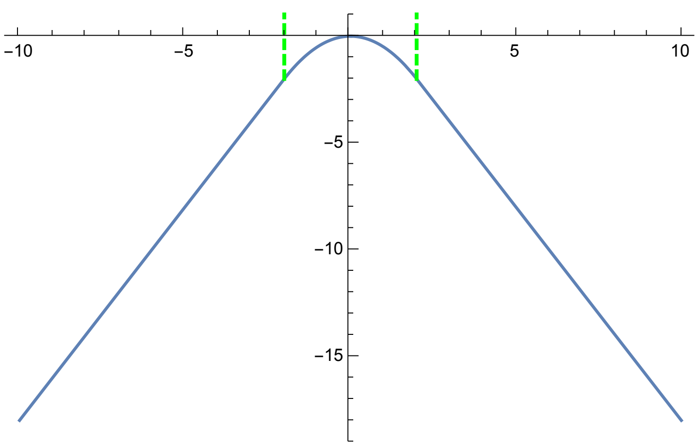
NOTE: y takes care of SIGNs; we use yprime as the variable of integration:

```
(* Input Cell - Enter your code here *)
ClearAll["`*"]
Ein[y_] = (rho y)/epsilonzero;
Eoutneg[y_] = -(rho a)/epsilonzero;
Eoutpos[y_] = (rho a)/epsilonzero;
Potin[y_] = -Integrate[Ein[yprime], {yprime, 0, y}]
Potoutpositive[y_] = Potin[a] - Integrate[Eoutpos[yprime], {yprime, a, y}]
Potoutnegative[y_] = Potin[-a] - Integrate[Eoutneg[yprime], {yprime, -a, y}]
Pottot[y_] =
  If[y < -a, Potoutnegative[y], If[-a ≤ y ≤ a, Potin[y], Potoutpositive[y]]];
rho = 1; epsilonzero = 1; a = 2;
PotPlot = Plot[Pottot[y], {y, -5 a, 5 a}];
lineleft = Graphics[{Thick, Dashed, Green, Line[{{-a, -a }, {-a, a }}]}];
lineright = Graphics[{Thick, Dashed, Green, Line[{{a, -a }, {a, a }}]}];
Show[PotPlot, lineleft, lineright]
```

$$-\frac{\text{rho}\, y^2}{2\,\text{epsilonzero}}$$

$$-\frac{a^2\,\text{rho}}{2\,\text{epsilonzero}}-\frac{a\,\text{rho}\,(-a+y)}{\text{epsilonzero}}$$

$$-\frac{a^2\,\text{rho}}{2\,\text{epsilonzero}}+\frac{a\,\text{rho}\,(a+y)}{\text{epsilonzero}}$$



(i)Comment on this result (the plot).

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The green dashed lines show the values of y (±a) at the edges of the slab.

The straight lines correspond to the <u>constant</u> E fields outside of the slab with slopes indicating the magnitudes of E

$$(\vec{E}[y] = -\, \text{Grad[Potential]} = -\, \frac{dV}{dy}\, \hat{y}).$$

Inside the slab, E changes linearly with y so that the potential $\sim -y^2$; this is consistent with the $\vec{E}$ pointing away from the center of the slab. Check yourself out on the SIGNS for the above and the directions of $\vec{E}$ in the various regions.