Real Inflation Dynamics, Path Integrals, and Macroeconomics

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"When i proposed the theory of relativity, very few understood me, and what i will reveal now to transmit to mankind will also collide with the misunderstanding and prejudice in the world. I ask you to guard the letters as long as necessary, years, decades, until society is advanced enough to accept what i will explain below. There is an extremely powerful force that, so far, science has not found a formal explanation to. It is a force that includes and governs all others, and is even behind any phenomenon operating in the universe and has not yet been identified by us. This universal force is LOVE."—From Albert Einstein to his daughter.

Abstract

This paper studies the time-varying transitional dynamics of the economy in the context of sticky prices and generalized hazard functions using a powerful new tool called "Path Integrals" that are especially effective in studying shocks to the economy that is far away from its steady state. In fact, there is no paper

*School of Economic Sciences, Washington State University at Pullman. I thank my advisors, Jinhui Bai, Salvador Ortigueira, and Jia Yan for their tireless guidance over time that has allowed me to pursue what i love and what i think was important in a right way. The world we are living in will be getting better with love rather than with hate or prejudice. I do believe generosity, empathy, the tolerance for mistakes and different ideas, and forgiveness all combined can be the one source of the real love that Einstein referred to. If we learn to forgive, there will be no hate and only love rules. I also thank the co-editor and the three anonymous referees for their thoughtful and informative suggestions and comments. Without my advisors, i would have never been able to have come thus far in the great field of macroeconomics. Without both my advisors and the co-editor and the three anonymous referees, this paper would have never existed or have already gone in a wrong direction very far and would never have a chance to take its current form that aims to maximize its long-run impact to improve the standard of living for human beings and deepen our understandings of macroeconomic dynamics.

studying transition dynamics following either an aggregate shock, a monetary shock, or an uncertainty shock whenever the distribution of interest is far away from the steady state, not to mention framing it in a setting of sticky prices and generalized hazard functions which we believe is more accurate to describe the real world. We will show in this paper that path integrals are an ideal way to study transition dynamics not only for sticky prices and inflation dynamics but also for many other settings in macroeconomics.

Key Words: Time-varying Inflation; Sticky Prices; Generalized Hazard Functions; Path Integrals; Impulse Response Function; Firm's Reinjection.

1 Introduction

Studying transition dynamics in macroeconomics is always tough, both analytically and computationally. We need to significantly deepen our understandings of macroeconomic dynamics and should not be complacent because technically speaking macroeconomics is more about dynamics than about statics. The following is basically how currently an average macroeconomic research is being done: they are either focused on the steady state of their models or transition dynamics that near the steady state. Currently, there is no better analytical approach to effectively study transition dynamics of macroeconomic models when the shocks hit the distribution of interest that is far away from the steady state. Analyzing historical data only is not going to help us better understand the underlying mechanisms of inflation dynamics and thus would certainly not help us accurately forecast the inflation dynamics in the future. Now, if we cannot accurately forecast future inflation dynamics over the transition because we have not yet fully understood the underlying theory of the inflation dynamics embedded in the economy, monetary policies made by central banks that aim to curb and stabilize inflation in the longer run would just be ineffective and as a consequence all consumers will pay the price. In general, we must admit that we just have not been able to develop a satisfying and good enough theoretical framework in macroeconomics yet to study a real transition dynamics.

Importantly, from the original work of Caballero and Engel (1993a, 1993b, 1999) as an example why studying transition dynamics that is far away from its steady state is so crucial, we see they were all pushing the idea that in these models the propagation of aggregate shocks depends on the distribution of the price gap which varies along the business cycle. Recent uncertain inflation dynamics and high interest rates as a consequence is another wake-up call for us to take seriously the transition dynamics that are not in anywhere near the steady state. We need to develop a better analytical and theoretical framework for macroeconomics to be able to effectively and accurately study the full range of the transition dynamics of the macroeconomy not only nearing the steady state but also being far away from the steady state.

To study the transition dynamics of the economy in a context of sticky prices, we take generalized hazard functions, which are the natural way of generalizing sticky prices set by firms that was originally developed by Caballero and Engel (1993a, 1993b), and further studied by Caballero and Engel (1999), Dotsey, King, and Wolman (1999), Woodford (2009) and Costain and Nakov (2011). The generalized hazard function has also been recently studied by Alvarez, Lippi, and Oskolkov (2022) and Alvarez and Lippi (2022) in which the sticky price model with both zero and steady-state inflation with the generalized hazard function has been systematically examined. This paper studies the time-varying transitional dynamics of the economy in the context of sticky prices and generalized hazard functions using a powerful new tool called "Path Integrals" that are especially effective in studying shocks to the economy that is far away from its steady state.

The contribution of this paper is as follows. First, path integrals allow us to get a fully analytical expression for the transition dynamics of the economy that not only near the steady state but also are far away from the steady state. As an application, we show how the effect of monetary shock following an uncertainty shock differs from the standard one-time monetary shock case using path integral formulation. Furthermore, we show that, by path integrals, the effect of a monetary shock following an uncertainty shock and the effect of a monetary shock followed by an uncertainty shock are quite different in terms of the transition dynamics. In other words, we show that the order of the shocks matters.

Second, in a context of sticky prices, we develop an approach to effectively address the impulse response of output to a monetary shock or a monetary shock following an uncertainty shock with firm's reinjection that is indispensable with sticky prices (i.e., with the generalized hazard functions) when the inflation is time-varying or non-zero constant rather than zero. This is the only approach in existing literature that can be used to deal with the firm's reinjection in sticky price settings and we view this advancement as groundbreaking.

Third, regarding our approach used for the analysis of the paper, our method of path integral formulation has its wide-ranging applications in macroeconomics far beyond the sticky-price models. For instance, path integral formulation also can help solve the aggregate dynamics in lumpy economies with time-dependent growth of aggregate productivity of the economy, which can be seen as a generalization of Baley and Blanco (2021) in which the growth of the aggregate productivity of their economy is a constant to a time-dependent growth of the aggregate productivity of the economy. A time-dependent growth of aggregate productivity of the economy can be viewed and modeled as an important driving force for any economy that is experiencing a transition from its initial steady state to a new steady state. It is our hope that both our framework of transitional inflation dynamics and our technical approach used to conduct the analysis will shed light on the future works in the related areas.

Indeed, the most exciting aspect of the framework of this paper is that it can be easily extended to study the transition dynamics of pretty much every model in macroeconomics associated with fixed adjustment cost and time-varying growth (i.e., whether it is time-varying productivity in lumpy investment, time-varying inflation in sticky price or time-varying average return in illiquid assets and so on), which are all impossible to analyze easily before, and from this perspective, the framework that this paper has helped to lay out encompasses macroeconomics with regard to its powerful capability to analyze the objects mentioned above.

1.1 The Setup in a Sticky Price Setting

The economic environment[1] considered in this paper is as follows. Consider an economy with a continuum of firms with the state variable of price gaps x_t subject to idiosyncratic risk and fixed price adjustment cost, where the state variable of the firm, namely, price gap, x_t, is the difference between the current charging price from the firm and an optimal frictionless charging price of the firm (i.e., the price that maximizes firm's profit with uncontrolled price gap process). The uncontrolled price gap x_t process is assumed to follow a continuous stochastic process, with generally time-varying drift $\mu(t)$ (the negative of $\mu(t)$ or $-\mu(t)$ stands for the time-varying

[1] I thank an anonymous referee for pointing out the good idea for the setup.

inflation) and constant volatility σ as

$$dx_t = \mu(t)dt + \sigma dW_t$$

where W_t is the Wiener process. Upon price adjustment, the firm will be sent back to the optimal point x^* (i.e., the point that maximizes firm's profit) at which the price gap is closed (i.e., at x^* the price gap is zero). We call the firm who is experiencing this returning to the optimal x^* right after its price adjustment as the reinjection of the firm. Dealing with reinjection of the firm is very challenging and no existing work has done just that. All the existing works are about models without considering firm's reinjection, which is appropriate only when the drift of the uncontrolled price process or inflation $-\mu(t)$ is zero, which will be justified below. Once $\mu(t)$ becomes non-zero or time-varying, which is exactly what this paper is focusing on, we have to consider firm's reinjection, otherwise, the model will be incorrect. This paper develops a groundbreaking approach to successfully help us obtain the impulse response function of output to a monetary shock with firm's reinjection in a context of time-varying inflation, sticky prices, and generalized hazard functions.

The price adjustment intensity in the setting of sticky prices is formulated using a "generalized hazard function" which generally, in the case of time-varying inflation $-\mu(t)$, can be written as $\Lambda(x,t)$ which is a function of both state variable x_t and time t. The generalized hazard function $\Lambda(x,t)$ determines the intensity of the price adjustment of the firm given state variable x_t and time t. This paper studies when $x \in (-\infty, \infty)$, i.e., the interval of state variable price gap x ranges from negative infinity to infinity or there is no boundary associated with the state variable.

The zero drift case $\mu(t) = 0$ with an implied generalized hazard function $\Lambda(x)$ is studied in various papers by Alvarez and Lippi (2022) and Alvarez, Lippi, and Oskolkov (2022), which yields a symmetric and stationary distribution of the price gaps and price changes.

The constant-drift case $\mu(t) = \mu$ with an implied generalized hazard function $\Lambda(x)$ is also studied by Alvarez, Lippi, and Oskolkov (2022), which yields an asymmetric and stationary distribution of price gaps and price changes. Moreover, the constant-drift case $\mu(t) = \mu$ with an implied piece-wise hazard is studied by Baley and Blanco (2021), which also yields an asymmetric and stationary environment.

In the previous two cases, there is a steady-state distribution of price gaps $\bar{p}(x)$ and price changes $\bar{q}(-x)$, which are both characterized by a time-independent Kolmogorov

Forward Equation (KFE). Its characterization is key for (i) studying the propagation of aggregate shocks (i.e., monetary shocks in the sticky-price setting and aggregate technology or productivity shocks in the lumpy investment setting), (ii) deriving sufficient statistics that characterize cumulative impulse response (CIR), and (iii) establishing mappings to the micro-data. This paper does not plan to explore the issue from the quantitative perspective but rather focuses on the theory part with applications of studying the effect of the monetary shock on the economy following an uncertainty shock to the initial stationary state of the economy in a context of time-varying inflation, sticky-price settings and the implied generalized hazard function, and leave a comprehensive quantitative exploration of the corresponding counterpart of this issue to a future work.

This paper first introduces path integrals to study the transition dynamics that are triggered by monetary shocks in a setting given above. Then, armed with path integrals, this paper considers a time-varying drift $\mu(t)$ with an implied generalized hazard function taking form of $\Lambda(x,t)$, which creates an interesting yet enormously challenging environment because of the implied non-stationarity that is caused by the time-varying drift $\mu(t)$ as well as the time-varying generalized hazard function $\Lambda(x,t)$. This paper explores the analytical characterization of the implied time-varying distribution of price gaps $p(x,t)$ and price changes $q(-x,t)$ using path integrals, a powerful tool which originates from quantum mechanics to solve the time-dependent KFE arising from the model covered in this paper which is impossible to solve by any other existing techniques, whether it is the standard eigenvalue-eigenfunction decomposition or the Laplace transform technique.

1.2 Impulse Response and Firm's Reinjection in Sticky Price Settings

The core idea for this paper is that we must differ a model without firm's reinjection from a model with firm's reinjection. In our context of sticky prices with generalized hazard functions and no bounds on state x, i.e., $-\infty < x < \infty$, the overall principle we must follow is as follows. If the drift or the inflation is zero and $-\infty < x < \infty$, then in terms of the impulse response of output to monetary shocks, a model with firm's reinjection is equivalent to a model without firm's reinjection on the ground that in such a case the implied generalized hazard functions are always symmetric around zero. By contrast, if the drift or the inflation is not zero and $-\infty < x < \infty$, no matter the drift or the inflation is a non-zero constant or a time-varying function, then in terms of the impulse response of output to monetary shocks, a model with firm's reinjection is not equivalent to a model without firm's reinjection on the ground that in this case the implied generalized hazard functions are no longer symmetric around zero anymore. Next, we explain why.

First, by firm' reinejction, we mean that we keep track of those firms, upon price adjustments, that are sent back to the optimal price x^* that closes up the price gap, so that the price gap for any firm right after their price adjustments is zero. In contrast, if we say a model without firm's reinjection, we mean that we do not keep track of those firms, upon price adjustments, that are sent back to the optimal price x^*. In a language of impulse response function of output to monetary shocks, the standard impulse response function with firm's reinjection, H of output, when $-\infty < x < \infty$, is defined as

$$H(t; -x, \delta, s) = \int_{-\infty}^{\infty} \mathbb{E}[-x(t)|x(0) = x]\big[p^{\delta,t}(x, s) - p^{0,t}(x, s)\big]dx \tag{1}$$

Here, several comments are in order. First, since the output is inversely proportional to price gap x, it follows that $-x(t)$ is the time path of output and hence the expectation is taken with respect to $-x(t)$. Second, at time $t = 0$, by definition, the state variable $x(0)$ is equal to the state variable itself x and this is why the expectation is conditional on $x(0) = x$. Third, $p^{\delta,t}(x, s)$ is the time s distribution of price gap x right after the monetary shock of size δ that occurs at time $t = s$ and $p^{0,t}(x, s)$ is the time s distribution of price gap x right before the monetary shock of size δ that occurs at time $t = s$. $p^{\delta,t}(x, s) - p^{0,t}(x, s)$ reflects the deviation of the distribution of price gap x triggered by the monetary shock of size δ that occurs at time $t = s$. Observe that by definition, $p^{\delta,t}(x, s) - p^{0,t}(x, s) = 0$ for all $s < t$, i.e., the distribution with and without the monetary shocks are equal before its arrival.

A related version of impulse response function G of output to monetary shocks without firm's reinjection is given by

$$G(t; -x, \delta, s) = \int_{-\infty}^{\infty} \mathbb{E}\big[-1_{\{0 \le t \le \tau\}} x(t)|x(0) = x\big]\big[p^{\delta,t}(x, s) - p^{0,t}(x, s)\big]dx \tag{2}$$

where τ is the time at which the firm makes its price adjustment. We see that the only difference between impulse response H with firm's reinjection and G without firm's reinjection is the indicator function that exists in the latter case which implies that in the latter case we do not keep track of those firms that have made their price adjustment at time $t = \tau$. In other words, in the case of G, we do not keep track of those firms when they are sent back to x^* right after the price adjustments.

Now, in the context of generalized hazard functions with $-\infty < x < \infty$, when the drift of the uncontrolled price gap process is zero, i.e., the inflation $\mu(t) = 0$, the implied generalized hazard functions are always symmetric around zero. For example, in a quadratic generalized hazard function case, the implied generalized hazard function with zero inflation takes the form $\Lambda(x) = \kappa x^2$ which is symmetric around zero and also implies the optimal $x^* = 0$. As a result, no matter whether or not we keep track of those firms that are sent back to the optimal x^* right after the price adjustments, the outputs of those firms right after their price adjustments at time $t = \tau$ are all equal to $-x(t = \tau) = -x^*(t = \tau) = 0$ so that it would not affect the expectation operators above which are taken with respect to the output. That is, the two expectations in H and G coincide with each other when the inflation is zero. Things will become much different if the drift or the inflation $\mu(t) \neq 0$, no matter it is a non-zero constant or a function of time t, the implied quadratic generalized hazard function would take the form $\Lambda(x, t) = \kappa \left[x - \frac{f(t)}{2\kappa} \right]^2$. We see that in this case the optimal $x^*(t) = \frac{f(t)}{2\kappa}$ which is not equal to zero, and therefore, in case H with firm's reinjection the expectation of $-x(t = \tau) = -x^*(t = \tau) = -\frac{f(\tau)}{2\kappa}$ which is not zero has to be taken into account, while in the case G the corresponding expectation is zero after $t = \tau$. Hence, H and G are different with non-zero inflation.

Given what we have discussed about the difference H and G, it does not imply that when $\mu(t) \neq 0$ or $\mu(t)$ is time-varying, the impulse response function G has no use at all. In fact, G plays a very significant role in determining H. Specifically, the impulse response function of H with firm's reinjection can be obtained through a summation over an infinite number of impulse response function of G with weight that accounts for the probabilistic occurrence of each term of the G in a context of treating the reinjections of firms as aggregate shocks for those firms who are experiencing returning to optimal point x^* right after their price adjustments. There is no other approach in existing literature from all subjects not just economics which can be used to help us get H, except for the approach we are going to introduce below. Therefore, as mentioned above, we view the approach and idea this paper develops for obtaining H as groundbreaking and very exciting. In what follows, let us describe this idea in more detail both intuitively and mathematically.

We first divide the time interval $t \in [0, \infty)$ into n arbitrary smaller time intervals with the end point for each segment being the arbitrary stopping time at which firms make price adjustments. Intuitively, what we are doing is that we assume there are n stopping times at which firms make their price adjustments. That is, τ_1 is the first time at which the firm makes its first price adjustment, τ_2 is the second time at which the firm makes its second price adjustment, and so on until τ_{n-1} is the $(n-1)$-th time at which the firm makes its $(n-1)$-th price adjustment. Those stopping times τs can be absolutely very arbitrary because we will show below that each arbitrary τ corresponds to a specific and unique probability for that τ to be a stopping time at which the firm makes its price adjustment. The divided time intervals thus follow the following pattern: $t \in [0, \tau_1]$, $t \in [\tau_1, \tau_2]$, $t \in [\tau_2, \tau_3]$, ...,$t \in [\tau_{n-1}, \tau_n)$. For each τ_1, τ_2, ..., τ_{n-1}, the firm makes price adjustment and thus returns to the optimal point x^* (i.e., the reinjection to the x^*). The novelty of the approach we are developing here is that we can view each reinjection as an aggregate shock for those we are returning to the x^*. Therefore, at each τ_1, τ_2, ..., τ_{n-1}, the initial distribution of the price gap x right before the reinjection or the "aggregate shock" is the distribution at each τ_1, τ_2, ..., τ_{n-1} as an right-end ending point of each interval, and the initial distribution of price gap x right after the reinjection or the "aggregate shock" is the distribution at each τ_1, τ_2, ..., τ_{n-1} as a left-end beginning point of each interval. Since for the reinjection, all the firms are sent back to the same point at a fixed stopping time, it follows that, by the fact that $x^*(t) = f(t)/2\kappa$ in the case of time-varying inflation $-\mu(t)$, the initial distribution of price gap x right after the reinjection at each τ_1, τ_2, ..., τ_{n-1} as a left-end beginning point of each interval are just Dirac delta function $\delta(f(\tau_1)/2\kappa)$, $\delta(f(\tau_2)/2\kappa)$, ..., $\delta(f(\tau_{n-1})/2\kappa)$, respectively. And the initial distributions of price gap x right before the reinjection at each τ_1, τ_2, ..., τ_{n-1} as an right-end ending point of each interval are the distributions, following time evolution of the distribution within that interval, fixed at the right ending time point.

Now, in each divided time interval, namely, $t \in [0, \tau_1]$, $t \in [\tau_1, \tau_2]$, $t \in [\tau_2, \tau_3]$, ...,$t \in [\tau_{n-1}, \tau_n)$, the impulse response function takes the form of G without firm's reinjection because the reinjection characterized as an "aggregate shock" for those who are experiencing returning to x^* occurs at the left-end point of each interval and therefore the effect of the reinjection can be characterized further by the impulse response function to the "aggregate shock" without firm's reinjection within each interval (since the firm's reinjection effect has already been replaced by the equivalent effect of aggregate shock). As a consequence, we have completely transformed the impulse response function H with firm's reinjection into the form of summation over impulse response function G without firm's reinjection with probabilistic occurrence. Mathematically, it is written as

$$\begin{aligned} H(t) = Pr(t \in [0, \tau_1])G(t \in [0, \tau_1]) + Pr(t \in [\tau_1, \tau_2])G(t \in [\tau_1, \tau_2]) + \\ \cdots + Pr(t \in [\tau_{n-2}, \tau_{n-1}])G(t \in [\tau_{n-2}, \tau_{n-1}]) + Pr(t \in [\tau_{n-1}, \tau_n])G(t \in [\tau_{n-1}, \tau_n]) \end{aligned} \tag{3}$$

or in a more compact form, $H(t)$ is rewritten in terms of $G(t)$ as

$$H(t) = \sum_{n=1}^{\infty} Pr(t \in [\tau_{n-1}, \tau_n])G(t \in [\tau_{n-1}, \tau_n]) \tag{4}$$

where we define $\tau_0 = 0$, $H(t)$ is the impulse response function with firm's reinjection, $G(t)$ is the impulse response function without firm's reinjection, and the $Pr(\cdot)$ is the corresponding probability for each of these τs to be the stopping time at which the firm makes price adjustment and in literature the $Pr(\cdot)$ is called survival function.

Finally, we can fully explore the impulse response function of output to a monetary shock or a monetary shock following an uncertainty shock with time-varying inflation, sticky prices, and generalized hazard functions with which the reinjection of firm has to be considered by utilizing impulse response function of $H(t)$ given by equation (4).

1.3 A Basic Motivation for Using Path Integrals

An intuitive motivation of the paper with path integrals rather than a standard approach (i.e., KFE formulation) is outlined as follows. Given the economic environment above, the standard KFE formulation of the problem when $\mu(t) = 0$ and generalized hazard function $\Lambda(x) = \kappa x^2$ without considering firm's reinjection is written as

$$\partial_t p(x,t) = (\sigma^2/2)\partial_x^2 p(x,t) - \kappa x^2 p(x,t).$$

There are several shortcomings associated with this standard KFE formulation for the problem above. First, this kind of KFE formulation does not allow for studying a sequence of shocks to the economy (i.e., a monetary shock following an uncertainty shock) in a compact form because KFE formulation assumes an initial condition right after the shock at time $t = 0$, i.e., $p(x, 0) = p_0(x)$. In other words, it does not assume any initial condition right after a shock at any point in time when $t \neq 0$ which is very important when there is a sequence of shocks where some of the shocks hit at $t \neq 0$ during the transition. Path integrals, by design, do not generally assume an initial condition $t = 0$ but rather $t = t_a$ where t_a can be any point in time. By this feature, path integrals provide us a very good way to study transition dynamics that are triggered not only by one-time shock but a sequence of shocks to the economy on the transition path at various points in time.

Second, KFE formulation only can give us an eigenvalue-eigenfunction solutions to the problem. While eigenvalue-eigenfunction solutions are useful for most of the analytical purposes, they sometimes are not very convenient for us to easily, promptly, and intuitively to get the full sense of the evolution of the transition because the eigenvalue-eigenfunction solutions are written in the forms of summation over infinite number of all the eigenvalues and eigenfunctions. Path integrals, as we will show throughout this paper, not only can be used to obtain eigenvalue-eigenfunction solutions but also solutions that are in the form that is easier for us to catch a quick sense of the transitional evolution intuitively.

Third, when the inflation becomes time-varying, i.e., $\mu(t)$ is not zero anymore but a function of real time t, the KFE formulation of the problem will be written in the form (first without firm's reinjection)

$$\partial_t p(x, t) = -\mu(t)\partial_x p(x, t) + (\sigma^2/2)\partial_x^2 p(x, t) - \Lambda(x, t)p(x, t),$$

and second with firm's reinjection as

$$\partial_t p(x, t) = -\mu(t)\partial_x p(x, t) + (\sigma^2/2)\partial_x^2 p(x, t) - \Lambda(x, t)p(x, t) + \Lambda(x, t)\delta(x(t) - x^*(t)),$$

where the last term $\Lambda(x, t)\delta(x(t) - x^*(t))$ captures the reinjections of the firms to the point $x^*(t)$ that maximizes their profits right after their price adjustment at rate $\Lambda(x, t)$ and $\delta(x(t) - x^*(t))$ is the Dirac Delta function at $x^*(t)$. The standard KFE formulation is not able to handle both cases in the sense that an analytical solution can be obtained. As we will show later in this paper, path integrals enable us to get a full analytical solution to such a problem with time-varying inflation implied by $\mu(t)$.

Above all, this paper is transformative for macroeconomics to study the transition dynamics in a following sense. Traditionally, Macroeconomists have to use partial differential equations, whether they are time-dependent Kolmogorov Forward Equations or Backward equations or any type of partial differential equations, to explore the transition dynamics of macroeconomy triggered by shocks not only just analytically but also computationally. Sometimes, it takes long time even for the computer to compute a particular partial differential equation, which is both expensive and not wise. The partial differential equations used by the macroeconomists to characterize the transition dynamics of macroeconomy are easy to formulate but always hard to analyze. For example, the time-dependent Kolmogorov Forward Equations are very easy to formulate based on any specific economic environment involving transition dynamics. However, once the the partial differential equations are written down, we would immediately find that it is not as easy to gain analytical insights from the partial differential equations as to write them down. In fact, in most cases, it is nearly impossible to get any analytical insights of the partial differential equations that are useful in analyzing the transition dynamics of the macroeconomy, because analytically analyzing partial differential equations is even hard for mathematicians, not to mention economists. We, as economists, need to take another approach.

This paper decisively abolishes the partial differential equations as a way to study macro dynamics and proposes path integrals instead to explore macroeconomic dynamics. Note that path integral formulation is not about the integral equations but rather the path integral itself. All it needs is to calculate the path integral, just like the way we calculate the traditional integral from elementary calculus classes. It is not anything relating to integral equations or differential equations. That is, we will not use any partial differential equation or integral equation in this paper to study the transition dynamics or any related object in our economic settings, and you will amazingly find that path integral formulation enables us to gain much more analytical insights into the macroeconomic dynamics than partial differential equations and integral equations combined could provide us.

2 Path Integrals and Transition Dynamics with Zero Inflation

2.1 Theoretical Framework

This section introduces path integrals in the context of sticky prices and generalized hazard functions to explore the transition dynamics of the economy. The basic idea about the path integrals for our economic settings of sticky prices and generalized hazard functions (whether it is for zero, constant, or time-varying inflation) is to analytically obtain the transition probability of price gap x in a time-varying transition process going from x_a at time t_a to x_b at time t_b. Here t_a and t_b can be any points in time during the time-varying transition process (especially, t_a has not necessarily to be the initial time, i.e., $t_a = t_0$, it can be any point in time during the transition). In fact, it is because t_a can be any point in time and not necessarily to be the initial time t_0 that real path integrals are the best tool available for us to analyze sequential multiple shocks which require multiple initial conditions at different points in time during the transition process. Throughout the paper, we call this transition probability of price gap x in a time-varying transition process going from x_a at time t_a to x_b at time t_b the Kernel denoted by $K(x_b, t_b; x_a, t_a)$.

Now, imagine a following thought experiment. First, let us denote any arbitrary fixed path of price gap z from time t_a at x_a to time t_b at x_b by $\bar{z}(t)$, and the actual paths of the price gap z from t_a to t_b by $z(t)$, where $t \in [t_a, t_b]$. Then, we can represent $z(t)$ in terms of $\bar{z}(t)$ and a new function $\hat{z}(t)$ so that

$$z(t) = \bar{z}(t) + \hat{z}(t) \tag{5}$$

that is, instead of defining a point on the path by its distance $z(t)$, we measure instead the deviation $\hat{z}(t)$ from the arbitrary fixed path $\bar{z}(t)$. The difference between the arbitrary fixed path $\bar{z}(t)$ and some possible alternative path $z(t)$ is the function $\hat{z}(t)$. Since the paths must both reach the same end points, it follows that $\hat{z}(t_a) = \hat{z}(t_b) = 0$.

In between these end points $\hat{z}(t)$ can take any form. Since the arbitrary fixed path is completely fixed, any variation in the alternative path $z(t)$ is equivalent to the associated variation in $\hat{z}(t)$. Thus, in a path integral, the path differential $\mathcal{D}z(t)$ can be replaced by $\mathcal{D}\hat{z}(t)$, i.e., $\mathcal{D}z(t) = \mathcal{D}\hat{z}(t)$, and the path $z(t)$ by $\bar{z}(t) + \hat{z}(t)$. Here, we use $\mathcal{D}$ to denote path differential rather than the ordinary differential d used in the standard calculus.

In this form, $\bar{z}(t)$ is a constant for the integration over the paths. Moreover, the new path variable $\hat{z}(t)$ is restricted to take the value 0 at both end points. This substitution leads to a path integral independent of end-point positions. (see Feynman and Hibbs, 1965). In what follows, we illustrate how to explore the transition dynamics of the economy due to a monetary shock by path integral formulation for the case where $\Lambda(x) = \kappa x^2$ and $\mu(t) = 0$ with volatility σ (i.e., the dynamics of the economy triggered by a monetary shock to the zero-inflation steady state with quadratic generalized hazard function $\Lambda(x) = \kappa x^2$ and volatility of the uncontrolled price process σ).

The real path integrals, given the above economic settings, are formulated by the following integral for the kernel $K(x_b, t_b; x_a, t_a)$ which represents the transition probability of price gap going from x_a at time t_a to x_b at time t_b as

$$K(x_b, t_b; x_a, t_a) = \int_{x_a}^{x_b} \exp\{-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left[\frac{1}{2}\dot{z}^2(\tau) + \sigma^2\Lambda(z)\right]d\tau\}\mathcal{D}z(\tau) \tag{6}$$

where z denotes the any possible transitional path of price gap x at time t_a and time t_b. $\mathcal{D}$ explicitly refers to the fact that the integral is taken with respect to all the possible paths of x between x_a and x_b (i.e., the path integrals not the usual integral denoted in this paper by d). Next proposition gives the explicitly computed version of the kernel $K(x_b, t_b; x_a, t_a)$ by path integrals techniques.

Proposition 1. *The kernel $K(x_b, t_b; x_a, t_a)$ that measures transition probability of state variable x, i.e., the price gap, going from x_a at time t_a to x_b at time t_b that was initially triggered by a monetary shock at time $t = t_a$ to a zero-inflation steady-state economy and an implied generalized hazard function $\Lambda(x) = \kappa x^2$ is given by*

$$\begin{aligned} K(x_b, t_b; x_a, t_a) = &\left[\frac{\sqrt{2\kappa}}{2\pi\sigma\sinh\sqrt{2\kappa}\sigma(t_b - t_a)}\right]^{1/2} \\ &\times \exp\left(-\frac{\sqrt{2\kappa}}{2\sigma}\left[\frac{(x_a^2 + x_b^2)\cosh\sigma\sqrt{2\kappa}(t_b - t_a) - 2x_a x_b}{\sinh\sigma\sqrt{2\kappa}(t_b - t_a)}\right]\right). \end{aligned} \tag{7}$$

The Proposition 1 implies that given the path integral formulation (2) of our problem, the calculated result for this formulation can be completely obtained analytically which is given by equation (3) of the Proposition 1. From the result, we see that the transition probability of price gap x that was initially triggered by a monetary shock to a zero-inflation steady-state economy with an implied generalized hazard function $\Lambda(x) = \kappa x^2$ is not written in terms of the eigenvalues and eigenfunctions but rather a completely algebraic form in a way that is more intuitive and easy to catch the insight of it. By contrast, the existing techniques (i.e., eigenvalue-eigenfunction decomposition technique) only can provide us the corresponding eigenvalue-eigenfunction solution which is a form of summation of infinite eigenvalues alongside eigenfunctions that makes the analytical result hard to follow intuitively. For example, we will provide the eigenvalue-eigenfunction solution as well below via path integrals. In fact, when dealing with firm's reinjection, we must use eigenvalue-eigenfunction solution rather than the more intuitive solution given by Proposition 1. However, when not dealing with firm's reinjection, we believe the more intuitive result given by Proposition 1 is a much better choice not only for its presentation but also for the applications.

Moreover, we see that the transition probability is written in terms of time frame from t_a to t_b and time t_a does not necessarily have to be t_0. In fact, t_a can be any point in time and not necessarily to be restricted by $t_a = t_0$. It is because of this property of the path integral formulation that studying a sequence of shocks to the economy in which some shocks may not happen at the initial time t_0 but rather at some later time generally denoted as t_a, where $t_a > t_0$, is much easier with path integrals rather than with any existing standard techniques which always assume an initial condition at time t_0 when the shock happens.

2.2 Applications

As a note at the beginning of applications in this section, since we study the transition dynamics of an economy with zero inflation, and based on our discussion earlier in the introduction that a model with zero inflation and with firm's reinjection is equivalent to the model with zero inflation and without firm's reinjection, it follows that we only need to study the model without firm's reinjection so that it covers both cases. Next, as applications of Proposition 1, we provide two exercises. We examine both the effect of one-time monetary shock of size δ on the transitional time path of density of price gap x and the effect of monetary shock of size δ following an uncertainty shock of size $\sigma - \sigma_0$ on the transitional time path of density of price gap x. For both exercises, we assume the initial steady-state density distribution of price gap x (that is, the steady-state density distribution of x before the monetary shock hits the economy) is normally distributed with mean 0 and variance $\frac{\sigma_0}{\sqrt{2\kappa}}$ which is the initial steady-state volatility. Actually, zero-inflation steady-state distribution of x with generalized hazard function κx^2 which is obviously symmetric around zero has to be with mean zero.

The reason for why assuming the initial stationary distribution of price gap x is normally distributed is largely due to its analytical convenience in the sense that the size of the monetary shock is actually equivalent to the change of the means of the two normal distributions of the price gaps which measures the difference of current charging price that always keeps fixed as long as price adjustment is not made and the profit-maximized price that is proportional to the cost of the firm before the shock and right after the shock. Provided what is given above, the initial steady-state density of x before the monetary shock is written as

$$p(x) = \left(\frac{\sqrt{2\kappa}}{\pi\sigma_0}\right)^{1/2} \exp\left(-\frac{\sqrt{2\kappa}}{\sigma_0}x^2\right) \tag{8}$$

We first consider the first exercise. For the first exercise, because there is only one one-time monetary shock of size δ, we would need to translate $x_a = y$, $t_a = 0$, $x_b = x$, and $t_b = t$ for more clarification and a better illustration. The original representation only better works for the sequence of shocks case which we will discuss later in the second exercise. Now, in the case of only one one-time monetary shock of size δ to the initial stationary distribution of price gap x at time $t = 0$, the initial distribution of the price gap x changes to the same type of normal distribution but with mean δ after the monetary shock of size δ.(Here we disregard whether it is a positive or negative monetary shock for convenience, only the magnitude of the monetary shock matters in our demonstration.) That is, right after the monetary shock, the distribution of the price gap x becomes

$$p(x) = \left(\frac{\sqrt{2\kappa}}{\pi\sigma_0}\right)^{1/2} \exp\left(-\frac{\sqrt{2\kappa}}{\sigma_0}(x-\delta)^2\right), \tag{9}$$

or using our more convenient notations introduced above for one-time shock case, it is rewritten as

$$p(y) = \left(\frac{\sqrt{2\kappa}}{\pi\sigma_0}\right)^{1/2} \exp\left(-\frac{\sqrt{2\kappa}}{\sigma_0}(y-\delta)^2\right), \tag{10}$$

Therefore, by the relationship between the transition probability and the density distribution, we have the following equation

$$p(x,t) = \int_{-\infty}^{\infty} K(x,t;y,0)p(y)dy \tag{11}$$

which says that the time path of the density distribution of x, $p(x,t)$, triggered by the monetary shock at time $t=0$ is equal to the inner product of the transition probability, $K(x,t;y,0)$, and the initial density distribution of x right after the monetary shock, $p(y)$.

Then, we proceed to get the formulation of time path of density distribution of price gap x, $p(x,t)$, by a direct substitution, as

$$\begin{aligned} p(x,t) = &\int_{-\infty}^{\infty} \left(\frac{\sqrt{2\kappa}}{2\pi\sigma_0 \sinh\sqrt{2\kappa}\sigma_0 t}\right)^{1/2} \exp\left(-\frac{\sqrt{2\kappa}}{2\sigma_0 \sinh\sqrt{2\kappa}\sigma_0 t}\left[\left(x^2+y^2\right)\cosh\sqrt{2\kappa}\sigma_0 t - 2xy\right]\right) \\ &\times \left(\frac{\sqrt{2\kappa}}{\pi\sigma_0}\right)^{1/2} \exp\left(-\frac{\sqrt{2\kappa}}{\sigma_0}(y-\delta)^2\right) dy \end{aligned} \tag{12}$$

Performing this Gaussian integral, we find the time path of density $p(x,t)$ is given by

$$\begin{aligned} p(x,t) = &\left(\frac{\sqrt{2\kappa}}{\pi\sigma_0 \cosh\sqrt{2\kappa}\sigma_0 t + 2\pi\sigma_0 \sinh\sqrt{2\kappa}\sigma_0 t}\right)^{1/2} \\ &\times \exp\left(-\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma_0 t\left(\cosh\sqrt{2\kappa}\sigma_0 t + 2\sinh\sqrt{2\kappa}\sigma_0 t\right) - 1\right]}{\sqrt{2\kappa}\sigma_0 \sinh\sqrt{2\kappa}\sigma_0 t\left(\cosh\sqrt{2\kappa}\sigma_0 t + 2\sinh\sqrt{2\kappa}\sigma_0 t\right)} x^2\right) \\ &\times \exp\left(\frac{4\kappa\delta}{\sigma_0\sqrt{2\kappa}\cosh\sqrt{2\kappa}\sigma_0 t + 2\sigma_0\sqrt{2\kappa}\sinh\sqrt{2\kappa}\sigma_0 t} x\right) \\ &\times \exp\left(\frac{4\kappa\delta^2 \sinh\sqrt{2\kappa}\sigma_0 t}{\sigma_0\sqrt{2\kappa}\cosh\sqrt{2\kappa}\sigma_0 t + 2\sigma_0\sqrt{2\kappa}\sinh\sqrt{2\kappa}\sigma_0 t} - \frac{\delta^2\sqrt{2\kappa}}{\sigma_0}\right) \end{aligned} \tag{13}$$

For concise notations, let

$$a(t) = \frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma_0 t\left(\cosh\sqrt{2\kappa}\sigma_0 t + 2\sinh\sqrt{2\kappa}\sigma_0 t\right) - 1\right]}{\sqrt{2\kappa}\sigma_0 \sinh\sqrt{2\kappa}\sigma_0 t\left(\cosh\sqrt{2\kappa}\sigma_0 t + 2\sinh\sqrt{2\kappa}\sigma_0 t\right)} > 0 \tag{14}$$

$$b(t) = \frac{4\kappa\delta}{\sigma_0\sqrt{2\kappa}\cosh\sqrt{2\kappa}\sigma_0 t + 2\sigma_0\sqrt{2\kappa}\sinh\sqrt{2\kappa}\sigma_0 t} \tag{15}$$

$$c(t) = \frac{4\kappa\delta^2\sinh\sqrt{2\kappa}\sigma_0 t}{\sigma_0\sqrt{2\kappa}\cosh\sqrt{2\kappa}\sigma_0 t + 2\sigma_0\sqrt{2\kappa}\sinh\sqrt{2\kappa}\sigma_0 t} - \frac{\delta^2\sqrt{2\kappa}}{\sigma_0} \tag{16}$$

and $p(x,t)$ can be concisely rewritten as

$$p(x,t) = \left(\frac{\sqrt{2\kappa}}{\pi\sigma_0\cosh\sqrt{2\kappa}\sigma_0 t + 2\pi\sigma_0\sinh\sqrt{2\kappa}\sigma_0 t}\right)^{1/2} \exp\left(-a(t)x^2 + b(t)x + c(t)\right) \tag{17}$$

A couple of comments are in order. First, note that the normalized $p(y, t = \tau)$ is written as

$$\begin{aligned} p(y, t=\tau) &= \frac{\exp\left(-a(\tau)y^2 + b(\tau)y + c(\tau)\right)}{\int_{-\infty}^{\infty}\exp\left(-a(\tau)y^2 + b(\tau)y + c(\tau)\right)dy} \\ &= \frac{\exp\left(-a(\tau)y^2 + b(\tau)y + c(\tau)\right)}{\sqrt{\frac{\pi}{a(\tau)}}\exp\left(\frac{b^2(\tau)}{4a(\tau)} + c(\tau)\right)} \\ &= \frac{\exp\left(-a(\tau)y^2 + b(\tau)y\right)}{\sqrt{\frac{\pi}{a(\tau)}}\exp\left(\frac{b^2(\tau)}{4a(\tau)}\right)} \\ &= \sqrt{\frac{a(\tau)}{\pi}}\exp\left(-a(\tau)\left[y - \frac{b(\tau)}{2a(\tau)}\right]^2\right) \end{aligned} \tag{18}$$

Second, given the time path of normalized density $p(x,t)$, we can easily derive the new invariant density $\bar{p}(x)$ by letting $t \to \infty$ in $p(x, t = \tau \to \infty)$, that is, $\bar{p}(x)$ is derived from the following expression

$$\bar{p}(x) = p(x, t = \tau \to \infty) = \lim_{\tau\to\infty}\frac{\exp\left(-a(\tau)x^2 + b(\tau)x + c(\tau)\right)}{\sqrt{\frac{\pi}{a(\tau)}}\exp\left(\frac{b^2(\tau)}{4a(\tau)} + c(\tau)\right)} \tag{19}$$

By the expressions of $a(\tau)$, $b(\tau)$, and $c(\tau)$ given above, we have

$$\begin{aligned} \lim_{\tau\to\infty} a(\tau) &= \lim_{\tau\to\infty}\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma_0\tau\left(\cosh\sqrt{2\kappa}\sigma_0\tau + 2\sinh\sqrt{2\kappa}\sigma_0\tau\right) - 1\right]}{\sqrt{2\kappa}\sigma_0\sinh\sqrt{2\kappa}\sigma_0\tau\left(\cosh\sqrt{2\kappa}\sigma_0\tau + 2\sinh\sqrt{2\kappa}\sigma_0\tau\right)} \\ &= \frac{\sqrt{2\kappa}}{2\sigma_0} \end{aligned} \tag{20}$$

$$
\begin{aligned}
\lim_{\tau\to\infty} b(\tau) &= \lim_{\tau\to\infty} \frac{4\kappa\delta}{\sigma_0\sqrt{2\kappa}\cosh\sqrt{2\kappa}\sigma_0\tau + 2\sigma_0\sqrt{2\kappa}\sinh\sqrt{2\kappa}\sigma_0\tau} \\
&= 0
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
\lim_{\tau\to\infty} c(\tau) &= \lim_{\tau\to\infty} \frac{4\kappa\delta^2\sinh\sqrt{2\kappa}\sigma_0\tau}{\sigma_0\sqrt{2\kappa}\cosh\sqrt{2\kappa}\sigma_0\tau + 2\sigma_0\sqrt{2\kappa}\sinh\sqrt{2\kappa}\sigma_0\tau} - \frac{\delta^2\sqrt{2\kappa}}{\sigma_0} \\
&= -\frac{\delta^2\sqrt{2\kappa}}{3\sigma_0}
\end{aligned}
\tag{22}
$$

Therefore, the new invariant density distribution of x, $\bar{p}(x)$ is given by

$$
\begin{aligned}
\bar{p}(x) &= \frac{\exp\left(-\frac{\sqrt{2\kappa}}{2\sigma_0}x^2 - \frac{\delta^2\sqrt{2\kappa}}{3\sigma_0}\right)}{\sqrt{\frac{\pi}{\sqrt{2\kappa}/2\sigma_0}}\exp\left(-\delta^2\sqrt{2\kappa}/3\sigma_0\right)} \\
&= \frac{\exp\left(-\frac{\sqrt{2\kappa}}{2\sigma_0}x^2\right)}{\sqrt{\frac{\pi}{\sqrt{2\kappa}/2\sigma_0}}} \\
&= \sqrt{\frac{\sqrt{2\kappa}/2\sigma_0}{\pi}}\exp\left(-\frac{\sqrt{2\kappa}}{2\sigma_0}x^2\right)
\end{aligned}
\tag{23}
$$

Now, we are ready to perform comparative statics to study the effect one-time monetary shock of size δ on the time evolution of distribution of price gap x, $p(x,t)$, as

$$
\frac{dp(x,t)}{d\delta} = \frac{d}{d\delta}\left[\sqrt{\frac{a(\tau)}{\pi}}\exp\left(-a(\tau)\left[y - \frac{b(\tau)}{2a(\tau)}\right]^2\right)\right] \tag{24}
$$

where

$$
a(\tau) = \frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma_0\tau\left(\cosh\sqrt{2\kappa}\sigma_0\tau + 2\sinh\sqrt{2\kappa}\sigma_0\tau\right) - 1\right]}{\sqrt{2\kappa}\sigma_0\sinh\sqrt{2\kappa}\sigma_0\tau\left(\cosh\sqrt{2\kappa}\sigma_0\tau + 2\sinh\sqrt{2\kappa}\sigma_0\tau\right)} \tag{25}
$$

$$
b(\tau) = \frac{4\kappa\delta}{\sigma_0\sqrt{2\kappa}\cosh\sqrt{2\kappa}\sigma_0\tau + 2\sigma_0\sqrt{2\kappa}\sinh\sqrt{2\kappa}\sigma_0\tau} \tag{26}
$$

$$c(\tau) = \frac{4\kappa\delta^2 \sinh \sqrt{2\kappa}\sigma_0\tau}{\sigma_0\sqrt{2\kappa}\cosh \sqrt{2\kappa}\sigma_0\tau + 2\sigma_0\sqrt{2\kappa}\sinh \sqrt{2\kappa}\sigma_0\tau} - \frac{\delta^2\sqrt{2\kappa}}{\sigma_0} \tag{27}$$

From this simple and intuitive expression of the effect of one-time monetary shock on the time evolution of price gap x, we can both intuitively and analytically learn how much effect and in what way and to what extent that the one-time monetary shock of size δ has played a role in shaping the long term evolution of the price-gap distribution of the economy in a way that no other existing technique could provide us.

Usually, price gaps are not as both intuitive and empirically observable as price changes. Price changes are the core concept in the inflation dynamics despite that the concept of price gaps and price changes are closely related and one can be easily derived from one other. Note that generalized hazard function with zero inflation $\Lambda(x)$ reflects the rate of the price change and therefore, given the time path of density of price gaps $p(x,t)$, the time path of density of price changes $q(-x,t)$, by the definition of the price changes is the negative of the price gaps, can be written as

$$q(-x,t) = \frac{\Lambda(x)p(x,t)}{\int_{-\infty}^{\infty} \Lambda(x)p(x,t)dx} \tag{28}$$

Finally, given the definition of price changes $q(-x,t)$, we can derive the effect of one-time monetary shock of size δ on the time path of density distribution of price changes $q(-x,t)$ with $\Lambda(x) = \kappa x^2$ as

$$\frac{dq(-x,t)}{d\delta} = \frac{d}{d\delta}\left[\frac{x^2p(x,t)}{\int_{-\infty}^{\infty} x^2p(x,t)dx}\right] \tag{29}$$

where $p(x,t)$ is given by

$$
\begin{aligned}
p(x,t) = &\left(\frac{\sqrt{2\kappa}}{\pi\sigma_0\cosh\sqrt{2\kappa}\sigma_0 t + 2\pi\sigma_0\sinh\sqrt{2\kappa}\sigma_0 t}\right)^{1/2} \\
&\times \exp\left(-\frac{\kappa\big[\cosh\sqrt{2\kappa}\sigma_0 t\big(\cosh\sqrt{2\kappa}\sigma_0 t + 2\sinh\sqrt{2\kappa}\sigma_0 t\big) - 1\big]}{\sqrt{2\kappa}\sigma_0\sinh\sqrt{2\kappa}\sigma_0 t\big(\cosh\sqrt{2\kappa}\sigma_0 t + 2\sinh\sqrt{2\kappa}\sigma_0 t\big)}x^2\right) \\
&\times \exp\left(\frac{4\kappa\delta}{\sigma_0\sqrt{2\kappa}\cosh\sqrt{2\kappa}\sigma_0 t + 2\sigma_0\sqrt{2\kappa}\sinh\sqrt{2\kappa}\sigma_0 t}x\right) \\
&\times \exp\left(\frac{4\kappa\delta^2\sinh\sqrt{2\kappa}\sigma_0 t}{\sigma_0\sqrt{2\kappa}\cosh\sqrt{2\kappa}\sigma_0 t + 2\sigma_0\sqrt{2\kappa}\sinh\sqrt{2\kappa}\sigma_0 t} - \frac{\delta^2\sqrt{2\kappa}}{\sigma_0}\right)
\end{aligned}
$$

We next consider the second exercise where we explore both scenarios: 1. the monetary shock occurs following an uncertainty shock, and 2. the uncertainty shock occurs following a monetary shock. We will show that these two scenarios give us two totally different effects of the shocks on the economy even for the overlapping part of the shocks. In other words, the order of the shocks matters, which has a huge policy implication on the effective timing of the monetary policy for example. (The non-overlap of the shocks has different effect on the economy is easy to understand just because they are different types of shocks with different sizes.) First, we consider a case where there is a monetary shock following an uncertainty shock. Imagine the following thought experiment. At time $t = 0$ an uncertainty shock happens by increasing the volatility up to σ so that the size of the uncertainty shock is equal to $\sigma - \sigma_0$, where $\sigma > \sigma_0$, because the initial steady-state volatility is at σ_0. Now, a monetary shock of size δ does not immediately follow in general until a time period of length of s has passed since the uncertainty shock of size $\sigma - \sigma_0$ at time $t = 0$. That is, the monetary shock of size δ following the uncertainty shock of size $\sigma - \sigma_0$ happens at time $t = s$ alongside the uncertainty shock of size $\sigma - \sigma_0$ at time $t = 0$. Our task is to ask what is the effect of the overall combined shocks on the time path of distribution of both price gaps and price changes? We should examine the combined effect in two steps obviously. First step is to determine the effect of the uncertainty shock only up until time $t = s$; and then second step is to view $t = s$ as a new initial time for incorporating the monetary shock into consideration. Therefore, in the first step, the effect of uncertainty shock from $t = 0$ until time $t = s$ can be thought of as a similar object as in our first exercise except that we only need to consider this first step until $t = s$ rather than forever. Given the initial uncertainty shock of size $\sigma - \sigma_0$ at time $t = 0$, the distribution of price gap right after this uncertainty shock at time $t = 0$ thus becomes (and note that at this moment there is no monetary shock yet)

$$p(y) = \left(\frac{\sqrt{2\kappa}}{\pi\sigma}\right)^{1/2} \exp\left(-\frac{\sqrt{2\kappa}}{\sigma} y^2\right) \tag{30}$$

Following the exact same procedure as in the first exercise, we find up to $t = s$ the time path of distribution of price gap $p(x,t)$ takes the form

$$\begin{aligned} p(x, t \le s) = \int_{-\infty}^{\infty} &\left(\frac{\sqrt{2\kappa}}{2\pi\sigma \sinh\sqrt{2\kappa}\sigma t}\right)^{1/2} \exp\left(-\frac{\sqrt{2\kappa}}{2\sigma\sinh\sqrt{2\kappa}\sigma t}\left[\left(x^2 + y^2\right)\cosh\sqrt{2\kappa}\sigma t - 2xy\right]\right) \\ &\times \left(\frac{\sqrt{2\kappa}}{\pi\sigma}\right)^{1/2} \exp\left(-\frac{\sqrt{2\kappa}}{\sigma} y^2\right) dy \end{aligned} \tag{31}$$

and the result is given by

$$\begin{aligned} p(x, t \le s) = &\left(\frac{\sqrt{2\kappa}}{\pi\sigma\cosh\sqrt{2\kappa}\sigma t + 2\pi\sigma\sinh\sqrt{2\kappa}\sigma t}\right)^{1/2} \\ &\times \exp\left(-\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma t\left(\cosh\sqrt{2\kappa}\sigma t + 2\sinh\sqrt{2\kappa}\sigma t\right) - 1\right]}{\sqrt{2\kappa}\sigma\sinh\sqrt{2\kappa}\sigma t\left(\cosh\sqrt{2\kappa}\sigma t + 2\sinh\sqrt{2\kappa}\sigma t\right)} x^2\right) \end{aligned} \tag{32}$$

In the second step from $t = s$ until forever, we consider the monetary shock that has been introduced into the economy. Two things happen. First, the monetary shock of size δ uniformly shifts the distribution $p(x, t \le s)$ at time $t = s$ so that the distribution right after the monetary shock of size δ at time $t = s$ without considering the sign of the monetary shock becomes

$$\begin{aligned} p(x, t = s) = &\left(\frac{\sqrt{2\kappa}}{\pi\sigma\cosh\sqrt{2\kappa}\sigma s + 2\pi\sigma\sinh\sqrt{2\kappa}\sigma s}\right)^{1/2} \\ &\times \exp\left(-\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s + 2\sinh\sqrt{2\kappa}\sigma s\right) - 1\right]}{\sqrt{2\kappa}\sigma\sinh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s + 2\sinh\sqrt{2\kappa}\sigma s\right)} (x - \delta)^2\right) \end{aligned} \tag{33}$$

Note that the normalized $p(y, t = s)$ is written as

$$p(y, t = s) = \sqrt{\frac{\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)-1\right]}{\sqrt{2\kappa}\sigma\sinh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)}}{\pi}}$$
$$\times\exp\left[-\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)-1\right]}{\sqrt{2\kappa}\sigma\sinh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)}(y-\delta)^2\right] \tag{34}$$

which is a normal distribution at time $t = s$ for the distribution of $p(y, t = s)$. Note that this way of normalization only works when $t = s$, i.e., when time is specifically fixed at a point in time.

Second, remember the kernel $K(x_b, t_b; x_a, t_a)$ that measures the transition probability of price gap x going from x_a at time t_a to x_b at time t_b. We just need to make a simple change of index for the initial time in order to cope with our current case. That is, let $t_a = s$. This is the reason we have repeatedly emphasized throughout that path integrals have led us to be better able to analytically handle a sequence of shocks that happen to the economy with some shocks happening not at the initial time $t = 0$. Therefore, starting right after the monetary shock of size δ at time $t = s$ until forever, the time evolution of the distribution of x can be expressed by the inner product of the kernel starting from $t = s$ and the distribution $p(x, t = s)$ as

$$p(x, t \geq s) = \int_{-\infty}^{\infty} K(x, t; y, s)p(y, t = s)dy \tag{35}$$

By a direct substitution, we get it re-expressed as

$$p(x, t \geq s) = \int_{-\infty}^{\infty}\left[\frac{\sqrt{2\kappa}}{2\pi\sigma\sinh\sqrt{2\kappa}\sigma(t-s)}\right]^{1/2}\exp\left(-\frac{\sqrt{2\kappa}}{2\sigma}\left[\frac{(y^2+x^2)\cosh\sigma\sqrt{2\kappa}(t-s)-2xy}{\sinh\sigma\sqrt{2\kappa}(t-s)}\right]\right)$$
$$\times\sqrt{\frac{\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)-1\right]}{\sqrt{2\kappa}\sigma\sinh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)}}{\pi}}$$
$$\times\exp\left[-\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)-1\right]}{\sqrt{2\kappa}\sigma\sinh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)}(y-\delta)^2\right]dy \tag{36}$$

and again the normalized time path of density of price changes with $\Lambda(x) = \kappa x^2$ is thus written as

$$q(-x,t) = \frac{x^2 p(x, t \geq s)}{\int_{-\infty}^{\infty} x^2 p(x, t \geq s) dx} \tag{37}$$

Next, as a comparison with the first scenario, we explore the second scenario with an opposite ordering of the shocks in which an uncertainty shock of size $\sigma - \sigma_0$ occurs at time $t = s$ following a monetary shock of size δ occurs at time $t = 0$. In such a circumstance, we conduct the following thought experiment which takes a similar fashion with the former scenario with a different ordering of the shocks. That is, at time $t = 0$ a monetary shock of size δ hits the steady-state economy. Now, an uncertainty shock of size $\sigma - \sigma_0$ does not immediately follow in general until a time period of length of s has passed since the initial monetary shock of size δ at time $t = 0$. That is, the uncertainty shock of size $\sigma - \sigma_0$ following the monetary shock of size δ happens at time $t = s$ alongside the monetary shock of size δ at time $t = 0$. Our task is to ask what is the effect of the overall combined shocks on the time path of distribution of both price gaps and price changes? We also should examine the combined effect in two steps obviously. First step is to determine the effect of the monetary shock only up until time $t = s$; and then second step is to view $t = s$ as a new initial time for incorporating the uncertainty shock into consideration. Therefore, in the first step, the effect of monetary shock from $t = 0$ until time $t = s$ can be thought of as a similar object as in our first exercise except that we only need to consider this first step until $t = s$ rather than forever. Given the initial monetary shock of size δ at time $t = 0$, the distribution of price gap right after this monetary shock at time $t = 0$ is again (note that here there is no uncertainty shock yet)

$$p(y) = \left(\frac{\sqrt{2\kappa}}{\pi\sigma_0}\right)^{1/2} \exp\left(-\frac{\sqrt{2\kappa}}{\sigma_0}(y-\delta)^2\right), \tag{38}$$

Following the exact same procedure as in the first exercise, we find up to $t = s$

the time path of distribution of price gap $p(x,t)$ takes the form

$$p(x,t\le s)=\int_{-\infty}^{\infty}\left(\frac{\sqrt{2\kappa}}{2\pi\sigma_0\sinh\sqrt{2\kappa}\sigma_0 t}\right)^{1/2}\exp\left(-\frac{\sqrt{2\kappa}}{2\sigma_0\sinh\sqrt{2\kappa}\sigma_0 t}\left[\left(x^2+y^2\right)\cosh\sqrt{2\kappa}\sigma_0 t-2xy\right]\right)$$
$$\times\left(\frac{\sqrt{2\kappa}}{\pi\sigma_0}\right)^{1/2}\exp\left(-\frac{\sqrt{2\kappa}}{\sigma_0}(y-\delta)^2\right)dy \tag{39}$$

and the result is given by

$$\begin{aligned}p(x,t\le s)=&\left(\frac{\sqrt{2\kappa}}{\pi\sigma_0\cosh\sqrt{2\kappa}\sigma_0 t+2\pi\sigma_0\sinh\sqrt{2\kappa}\sigma_0 t}\right)^{1/2}\\&\times\exp\left(-\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma_0 t\left(\cosh\sqrt{2\kappa}\sigma_0 t+2\sinh\sqrt{2\kappa}\sigma_0 t\right)-1\right]}{\sqrt{2\kappa}\sigma_0\sinh\sqrt{2\kappa}\sigma_0 t\left(\cosh\sqrt{2\kappa}\sigma_0 t+2\sinh\sqrt{2\kappa}\sigma_0 t\right)}x^2\right)\\&\times\exp\left(\frac{4\kappa\delta}{\sigma_0\sqrt{2\kappa}\cosh\sqrt{2\kappa}\sigma_0 t+2\sigma_0\sqrt{2\kappa}\sinh\sqrt{2\kappa}\sigma_0 t}x\right)\\&\times\exp\left(\frac{4\kappa\delta^2\sinh\sqrt{2\kappa}\sigma_0 t}{\sigma_0\sqrt{2\kappa}\cosh\sqrt{2\kappa}\sigma_0 t+2\sigma_0\sqrt{2\kappa}\sinh\sqrt{2\kappa}\sigma_0 t}-\frac{\delta^2\sqrt{2\kappa}}{\sigma_0}\right)\end{aligned}$$

In the second step from $t=s$ until forever, we consider the uncertainty shock that has been introduced into the economy. Two things happen. First, the uncertainty shock of size $\sigma-\sigma_0$ changes the parameter of σ_0 appearing in the distribution $p(x,t\le s)$ to σ at time $t=s$ so that the distribution right after the uncertainty shock of size $\sigma-\sigma_0$ at time $t=s$ becomes

$$\begin{aligned}p(y,t=s)=&\left(\frac{\sqrt{2\kappa}}{\pi\sigma\cosh\sqrt{2\kappa}\sigma s+2\pi\sigma\sinh\sqrt{2\kappa}\sigma s}\right)^{1/2}\\&\times\exp\left(-\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)-1\right]}{\sqrt{2\kappa}\sigma\sinh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)}y^2\right)\\&\times\exp\left(\frac{4\kappa\delta}{\sigma\sqrt{2\kappa}\cosh\sqrt{2\kappa}\sigma s+2\sigma\sqrt{2\kappa}\sinh\sqrt{2\kappa}\sigma s}y\right)\\&\times\exp\left(\frac{4\kappa\delta^2\sinh\sqrt{2\kappa}\sigma s}{\sigma\sqrt{2\kappa}\cosh\sqrt{2\kappa}\sigma s+2\sigma\sqrt{2\kappa}\sinh\sqrt{2\kappa}\sigma s}-\frac{\delta^2\sqrt{2\kappa}}{\sigma}\right)\end{aligned}$$

Note that the normalized $p(y,t=s)$ is written as

$$p(y,t=s) = \frac{\exp\left(-a(s)y^2+b(s)y+c(s)\right)}{\int_{-\infty}^{\infty}\exp\left(-a(s)y^2+b(s)y+c(s)\right)dy}$$
$$= \frac{\exp\left(-a(s)y^2+b(s)y+c(s)\right)}{\sqrt{\frac{\pi}{a(s)}}\exp\left(\frac{b^2(s)}{4a(s)}+c(s)\right)}$$
$$= \frac{\exp\left(-a(s)y^2+b(s)y\right)}{\sqrt{\frac{\pi}{a(s)}}\exp\left(\frac{b^2(s)}{4a(s)}\right)}$$
$$= \sqrt{\frac{a(s)}{\pi}}\exp\left(-a(s)\left[y-\frac{b(s)}{2a(s)}\right]^2\right) \tag{40}$$

where

$$a(s) = \frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s + 2\sinh\sqrt{2\kappa}\sigma s\right)-1\right]}{\sqrt{2\kappa}\sigma\sinh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)} > 0 \tag{41}$$

$$b(s) = \frac{4\kappa\delta}{\sigma\sqrt{2\kappa}\cosh\sqrt{2\kappa}\sigma s + 2\sigma\sqrt{2\kappa}\sinh\sqrt{2\kappa}\sigma s} \tag{42}$$

$$c(s) = \frac{4\kappa\delta^2\sinh\sqrt{2\kappa}\sigma s}{\sigma\sqrt{2\kappa}\cosh\sqrt{2\kappa}\sigma s + 2\sigma\sqrt{2\kappa}\sinh\sqrt{2\kappa}\sigma s} - \frac{\delta^2\sqrt{2\kappa}}{\sigma} \tag{43}$$

which is a normal distribution at time $t=s$ for the distribution of $p(y,t=s)$. Note again that this way of normalization only works when $t=s$, i.e., when time is specifically fixed at a point in time.

Second, again remember the kernel $K(x_b,t_b;x_a,t_a)$ that measures the transition probability of price gap x going from x_a at time t_a to x_b at time t_b. We just need to make a simple change of index for the initial time in order to cope with our current case. That is, let $t_a=s$ and change σ_0 in the kernel to σ due to the uncertainty shock of size $\sigma-\sigma_0$ introduced into the economy at time $t=s$. Therefore, starting right after the uncertainty shock of size $\sigma-\sigma_0$ at time $t=s$ until forever, the time evolution of the distribution of x can be expressed by the inner product of the kernel starting from $t=s$ and the distribution $p(x,t=s)$ as

$$p(x,t\geq s) = \int_{-\infty}^{\infty}K(x,t;y,s)p(y,t=s)dy \tag{44}$$

By a direct substitution, we get it re-expressed as

$$p(x,t\geq s)=\int_{-\infty}^{\infty}\left[\frac{\sqrt{2\kappa}}{2\pi\sigma\sinh\sqrt{2\kappa}\sigma(t-s)}\right]^{1/2}e^{\left(-\frac{\sqrt{2\kappa}}{2\sigma}\left[\frac{\left(y^2+x^2\right)\cosh\sigma\sqrt{2\kappa}(t-s)-2xy}{\sinh\sigma\sqrt{2\kappa}(t-s)}\right]\right)}$$
$$\times\sqrt{\frac{\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)-1\right]}{\sqrt{2\kappa}\sigma\sinh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)}}{\pi}}$$
$$\times e^{\left(-\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)-1\right]}{\sqrt{2\kappa}\sigma\sinh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)}\left[y-\frac{\frac{4\kappa\delta}{\sigma\sqrt{2\kappa}\cosh\sqrt{2\kappa}\sigma s+2\sigma\sqrt{2\kappa}\sinh\sqrt{2\kappa}\sigma s}}{2\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)-1\right]}{\sqrt{2\kappa}\sigma\sinh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)}}\right]^2\right)}dy \tag{45}$$

and again the normalized time path of density of price changes with $\Lambda(x)=\kappa x^2$ is thus written as

$$q(-x,t)=\frac{x^2p(x,t\geq s)}{\int_{-\infty}^{\infty}x^2p(x,t\geq s)dx} \tag{46}$$

If we compare the time evolution of distribution of price gap $p(x,t\geq s)$ with monetary shock following the uncertainty shock and the same $p(x,t\geq s)$ but with uncertainty shock following the monetary shock, we find they (i.e., the overlap of the shocks happens when $t\geq s$) are quite different. For a better differentiation, we use $p^{um}(x,t\geq s)$ to denote the former case and use $p^{mu}(x,t\geq s)$ to denote the latter case, then we rewrite the two below for a better comparison:

$$p^{um}(x,t\geq s)=\int_{-\infty}^{\infty}\left[\frac{\sqrt{2\kappa}}{2\pi\sigma\sinh\sqrt{2\kappa}\sigma(t-s)}\right]^{1/2}\exp\left(-\frac{\sqrt{2\kappa}}{2\sigma}\left[\frac{(y^2+x^2)\cosh\sigma\sqrt{2\kappa}(t-s)-2xy}{\sinh\sigma\sqrt{2\kappa}(t-s)}\right]\right)$$
$$\times\sqrt{\frac{\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)-1\right]}{\sqrt{2\kappa}\sigma\sinh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)}}{\pi}}$$
$$\times\exp\left[-\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)-1\right]}{\sqrt{2\kappa}\sigma\sinh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)}(y-\delta)^2\right]dy \tag{47}$$

$$p^{mu}(x,t \geq s) = \int_{-\infty}^{\infty} \left[\frac{\sqrt{2\kappa}}{2\pi\sigma \sinh \sqrt{2\kappa}\sigma(t-s)} \right]^{1/2} e^{\left(-\frac{\sqrt{2\kappa}}{2\sigma} \left[\frac{\left(y^2+x^2\right)\cosh \sigma\sqrt{2\kappa}(t-s) - 2xy}{\sinh \sigma\sqrt{2\kappa}(t-s)} \right] \right)}$$
$$\times \sqrt{\frac{\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)-1\right]}{\sqrt{2\kappa}\sigma\sinh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)}}{\pi}}$$
$$\times e^{\left(-\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)-1\right]}{\sqrt{2\kappa}\sigma\sinh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)} \left[y - \frac{\frac{4\kappa\delta}{\sigma\sqrt{2\kappa}\cosh\sqrt{2\kappa}\sigma s+2\sigma\sqrt{2\kappa}\sinh\sqrt{2\kappa}\sigma s}}{2\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)-1\right]}{\sqrt{2\kappa}\sigma\sinh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)}} \right]^2 \right)} dy \tag{48}$$

By a direct comparison, we find that $p^{um}(x,t \geq s)$ is not equal to $p^{mu}(x,t \geq s)$. That is, $p^{um}(x,t \geq s) \neq p^{mu}(x,t \geq s)$, because in general, we have

$$\delta \neq \frac{\frac{4\kappa\delta}{\sigma\sqrt{2\kappa}\cosh\sqrt{2\kappa}\sigma s+2\sigma\sqrt{2\kappa}\sinh\sqrt{2\kappa}\sigma s}}{2\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)-1\right]}{\sqrt{2\kappa}\sigma\sinh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)}} \tag{49}$$

or equivalently,

$$1 \neq \frac{\frac{4\kappa}{\sigma\sqrt{2\kappa}\cosh\sqrt{2\kappa}\sigma s+2\sigma\sqrt{2\kappa}\sinh\sqrt{2\kappa}\sigma s}}{2\frac{\kappa\left[\cosh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)-1\right]}{\sqrt{2\kappa}\sigma\sinh\sqrt{2\kappa}\sigma s\left(\cosh\sqrt{2\kappa}\sigma s+2\sinh\sqrt{2\kappa}\sigma s\right)}} \tag{50}$$

How and to what extend these two objects differ from each other, based on the expressions above, turns out to be completely dependent only on two parameters: one is the duration of time lapse s between the first shock and the following shock, regardless of the ordering of the shocks, and another one is the size of the new volatility σ of the economy. Interestingly, we see that the difference of the two does not depend on first, the size of the monetary shock of size δ, and second, the initial volatility σ_0, which implies that the monetary shock of size δ and the initial uncertainty of the economy σ_0 does not contribute to the difference of the combined effects for the transition dynamics of the economy triggered by the combined shocks with different ordering of the shocks. In general, we say that the time evolution of distribution of price gap and price change derived from the monetary shock following an uncertainty shock is not the same as the time evolution of distribution of price gap and price change derived from the uncertainty shock following a monetary shock. Once again, we see the power of path integrals has offered us in studying the transition dynamics of macroeconomy in a way that no existing technique has been even close to offering us.

It is also from this result that we see the traditional way of analyzing transition dynamics with a sequence of shocks using linearization should be very wrong. First, linearization will directly lead to the overall combined effects of the sequence of shocks being equalized to each other even with a different ordering of the shocks. Second, even more importantly, linearization could not allow us to even effectively study a shock that is far away from the steady state because linearization only works for transition dynamics near the steady state. Path integrals, as we have seen throughout up to this point, provide us perfect tools to study the transition dynamics even with the state of the economy is far away from the initial steady state as we have discussed above.(Remember t_a is effectively useful across all time.)

In what follows as another application exercise from a different perspective but in the same economic setting, we explore the effect of uncertainty shock on the impulse response of output to a monetary shock. Indeed, as Alvarez and Lippi (2022) argue that studying the effect that changes to the volatility of shocks exert on the propagation of monetary shocks matters to the effectiveness of monetary policy in recession versus boom, when the state of the economy is assumed to feature, respectively, high versus low volatility of shocks. Alvarez and Lippi (2022) study such an issue with constant rate of price adjustment, or in other words, without generalized hazard function as a function of the state variable to be the rate of the price adjustment. In this paper, we aim to study a similar object with generalized hazard function via path integrals, which substantially extends their analysis to the state-dependent rate of price adjustment.

Recall that the impulse response function of output, without firm's reinjection, to monetary shocks, is given by

$$G(t;-x,\delta,s)=\int_{-\infty}^{\infty}\mathbb{E}\big[-1_{\{t\leq\tau\}}x(t)|x(0)=x\big]\big[p^{\delta,t}(x,s)-p^{0,t}(x,s)\big]dx$$

But for this paper we consider the cumulative version of the impulse response $M(\delta,s)$ which by definition measures the area enclosed by the impulse response function and the time axis written as

$$M(\delta, s) = \int_{-\infty}^{\infty} \mathbb{E}\Big[-1_{\{t\leq\tau\}} \int_{s}^{\infty} x(t)dt|x(0) = x\Big] \big[p^{\delta,t}(x, s) - p^{0,t}(x, s)\big]dx$$

or equivalently,

$$M(\delta, s) = \int_{s}^{\infty} \int_{-\infty}^{\infty} \mathbb{E}\big[-1_{\{t\leq\tau\}}x(t)|x(0) = x\big] \big[p^{\delta,s}(x, t) - p^{0,s}(x, t)\big]dxdt$$

where we define $\mathcal{M}(x, t) = \mathbb{E}\big[-1_{\{t\leq\tau\}}x(t)|x(0) = x\big]$.

Note that since whenever $t < s$, M is zero because $p^{\delta,t}(x, s) - p^{0,t}(x, s) = 0$ when $t < s$, that is when the monetary shock has not arrived, and hence we consider the cumulative impulse response from s until forever. There two different components included in this formulation of impulse response function M, namely, the expectation component $\mathbb{E}\big[-1_{\{t\leq\tau\}}x(t)|x(0) = x\big]$ and the density distribution component $p^{\delta,t}(x, s) - p^{0,t}(x, s)$. We then need to figure out both components analytically by path integrals. Armed with the Proposition 1 that offers the kernel $K(x_b, t_b; x_a, t_a)$, both of the components are easy to present analytically in the context of monetary shock following an uncertainty shock. Assume at time $t = 0$ an uncertainty shock of size $\sigma - \sigma_0$ takes place, which changes the initial steady-state volatility of the economy from σ_0 to σ and the transition dynamics of the economy begins at time $t = 0$ until at time $t = s$ when there is another monetary shock of size δ hitting the economy and joining the initial uncertainty shock to affect the transition dynamics of the economy. Then, our question is that what is the effect of the uncertainty shock of size $\sigma - \sigma_0$ on the impulse response of the output to the monetary shock of size δ. Recall that the initial stationary distribution of price gap $p(x)$ before any shock is given by

$$p(x) = \left(\frac{\sqrt{2\kappa}}{\pi\sigma_0}\right)^{1/2} \exp\left(-\frac{\sqrt{2\kappa}}{\sigma_0}x^2\right)$$

and the kernel K is given by

$$\begin{aligned} K(x_b, t_b; x_a, t_a) = &\left[\frac{\sqrt{2\kappa}}{2\pi\sigma \sinh \sqrt{2\kappa}\sigma(t_b - t_a)}\right]^{1/2} \\ &\times \exp\left(-\frac{\sqrt{2\kappa}}{2\sigma}\left[\frac{(x_a^2 + x_b^2)\cosh \sigma\sqrt{2\kappa}(t_b - t_a) - 2x_a x_b}{\sinh \sigma\sqrt{2\kappa}(t_b - t_a)}\right]\right) \end{aligned}$$

Now, with those two important elements, namely, the initial stationary distribution of state and kernel that characterizes the transition probability of the dynamics, let us figure out the both components in the two phases, i.e., the first phase of only uncertainty shock of size $\sigma - \sigma_0$ from $t = 0$ to $t = s$ and the second phase of both the initial uncertainty shock of size $\sigma - \sigma_0$ and the monetary shock of size δ from $t = s$ until forever. At time $t = 0$, an uncertainty shock of size $\sigma - \sigma_0$ occurs, so that two things happen. First, the initial stationary distribution of state becomes

$$p(x) = \left(\frac{\sqrt{2\kappa}}{\pi\sigma}\right)^{1/2} \exp\left(-\frac{\sqrt{2\kappa}}{\sigma}x^2\right) \tag{51}$$

and the kernel from $t = 0$ to $t = s$ is given by exactly the same above. Therefore, the distribution of the state at time $t = s$ before the monetary shock is given by $p^{0,t}(x, s)$

$$p^{0,t}(x, s) = \int_{-\infty}^{\infty} K(x, s; y, 0; \sigma) \left(\frac{\sqrt{2\kappa}}{\pi\sigma}\right)^{1/2} \exp\left(-\frac{\sqrt{2\kappa}}{\sigma}y^2\right) dy \tag{52}$$

and the the corresponding distribution of state at time s right after the monetary shock of size δ thus becomes

$$p^{\delta,t}(x, s) = \int_{-\infty}^{\infty} K(x - \delta, s; y, 0; \sigma) \left(\frac{\sqrt{2\kappa}}{\pi\sigma}\right)^{1/2} \exp\left(-\frac{\sqrt{2\kappa}}{\sigma}y^2\right) dy \tag{53}$$

The second component $\mathcal{M}(x, t) = \mathbb{E}\big[-1_{\{t\leq\tau\}}x(t)|x(0) = x\big]$ can be solved by eigenvalue-eigenfunction solution with via path integrals and the solution is given by

$$\begin{aligned}\mathcal{M}(x, t) = &-\left(\frac{\sqrt{\kappa}}{\pi\sigma}\right)^{1/2} e^{-\left(\frac{\sqrt{\kappa}}{2\sigma}\right)x^2} \\ &\times \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(2^n n!)} H_n\left(\sqrt{\frac{\sqrt{\kappa}}{\sigma}}x\right) H_n\left(\sqrt{\frac{\sqrt{\kappa}}{\sigma}}y\right) e^{-\left(\frac{\sqrt{\kappa}}{2\sigma}\right)y^2} y e^{-\sqrt{\kappa}\sigma\left(n+\frac{1}{2}\right)t} dy\end{aligned} \tag{54}$$

where $H_n(\cdot)$ is the Hermite polynomials.

Putting the two components together, we get the cumulative version of the impulse response written as

$$
\begin{aligned}
M(\delta,\sigma,s) = & \int_s^\infty \int_{-\infty}^\infty -\left(\frac{\sqrt{\kappa}}{\pi\sigma}\right)^{1/2} e^{-\left(\frac{\sqrt{\kappa}}{2\sigma}\right)x^2} \\
& \times \sum_{n=1}^\infty \int_{-\infty}^\infty \frac{1}{(2^n n!)} H_n\left(\sqrt{\frac{\sqrt{\kappa}}{\sigma}}x\right) H_n\left(\sqrt{\frac{\sqrt{\kappa}}{\sigma}}y\right) e^{-\left(\frac{\sqrt{\kappa}}{2\sigma}\right)y^2} y e^{-\sqrt{\kappa}\sigma\left(n+\frac{1}{2}\right)t} dy \\
& \times \int_{-\infty}^\infty K(x-\delta,s;y,0;\sigma)\left(\frac{\sqrt{2\kappa}}{\pi\sigma}\right)^{1/2} \exp\left(-\frac{\sqrt{2\kappa}}{\sigma}y^2\right) dydxdt \\
& - \int_s^\infty \int_{-\infty}^\infty -\left(\frac{\sqrt{\kappa}}{\pi\sigma}\right)^{1/2} e^{-\left(\frac{\sqrt{\kappa}}{2\sigma}\right)x^2} \\
& \times \sum_{n=1}^\infty \int_{-\infty}^\infty \frac{1}{(2^n n!)} H_n\left(\sqrt{\frac{\sqrt{\kappa}}{\sigma}}x\right) H_n\left(\sqrt{\frac{\sqrt{\kappa}}{\sigma}}y\right) e^{-\left(\frac{\sqrt{\kappa}}{2\sigma}\right)y^2} y e^{-\sqrt{\kappa}\sigma\left(n+\frac{1}{2}\right)t} dy \\
& \times \int_{-\infty}^\infty K(x,s;y,0;\sigma)\left(\frac{\sqrt{2\kappa}}{\pi\sigma}\right)^{1/2} \exp\left(-\frac{\sqrt{2\kappa}}{\sigma}y^2\right) dydxdt
\end{aligned}
\tag{55}
$$

where $K(x,s;y,0;\sigma)$ is given by

$$
K(x,s;y,0;\sigma) = \left[\frac{\sqrt{2\kappa}}{2\pi\sigma\sinh\sqrt{2\kappa}\sigma s}\right]^{1/2} \exp\left(-\frac{\sqrt{2\kappa}}{2\sigma}\left[\frac{(x^2+y^2)\cosh\sigma\sqrt{2\kappa}s - 2xy}{\sinh\sigma\sqrt{2\kappa}s}\right]\right)
\tag{56}
$$

A simple comparative static exercise gives us the effect of uncertainty shock of size σ on the cumulative version of the impulse response of output to a monetary shock of size δ as $\frac{dM(\delta,\sigma,s)}{d\sigma}$, where $M(\delta,\sigma,s)$ is given by equation (55).

3 Path Integrals and Inflation Dynamics

This section extends section 2 where $\mu(t) = 0$ to a case in which $\mu(t)$ is time-varying (i.e., inflation is time-varying) using path integrals. We first studies the corresponding firm's problem with time-varying inflation which rationalizes the functional form of generalized hazard function $\Lambda(x,t)$ in such a case. We then study the impulse response function of output to a monetary shock and a monetary shock following an uncertainty shock with time-varying inflation, sticky prices, and generalized hazard function. Note that, unlike section 2 where $\mu(t) = 0$ with which the impulse response functions with and without firm's reinjection are the same, in a case of time-varying $\mu(t)$, we must use impulse response function $H(t)$ that is an impulse response function with firm's reinjection rather than $G(t)$ which is an impulse response function without firm's reinjection. Luckily, by the foundation we have developed in the introduction that $H(t)$ can be basically written as a linear combination of $G_n(t)$ with probabilistic occurrence for each $G_n(t)$, we are capable of exploring $H(t)$ in this section with time-varying inflation.

3.1 Firm's Problem

First, we study the firm's problem with time-varying inflation. In the firm's problem with time-varying inflation $\mu(t)$, the generalized hazard function is $\Lambda(x, t)$ which is a map from the state variable x to the rate of price adjustment Λ over the transitional period. The firm can change its price only by paying a flow cost $c(l(x, t))$ at each moment to obtain a free opportunity at an arrival rate $l(x, t)$ to make the price adjustment. Assume the flow cost $c(l(x, t))$ is increasing and convex in l. Unlike Alvarez, Lippi, and Oskolkov (2022), we do not allow the firm to pay a deterministic menu cost Ψ to change its price with certainty. Or put it in another way, we assume $\Psi = \infty$ so that there will be no finite barriers $(\underline{x}, \overline{x})$ for the firm to change the price once it hits the barriers. In other words, in our case $\underline{x} = -\infty$ and $\overline{x} = \infty$, where x denote the difference between the optimal profit-maximizing price and the actual charging price of the firm or so-called price gap. When there is no price adjustment occurring, we assume the price gap x evolves according to (i.e., uncontrolled price process with time-varying inflation)

$$dx(t) = \mu(t)dt + \sigma dW(t) \tag{57}$$

where $\mu(t)$ represents the negative value of the transitional or time-dependent inflation of the economy, σ is the idiosyncratic volatility of the price shocks, and $W(t)$ is the standard Brownian motion. We also assume firm's per period profit function is $-Bx^2$ and a transitional interest rate $r(t)$. Then, the firm's problem can be written as a HJB equation as

$$
\begin{aligned}
r(t)v(x,t) = Bx^2 + \mu(t)\frac{\partial v(x,t)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 v(x,t)}{\partial x^2} \\
+ \min_{l\geq 0}\left\{l(x,t)(v(x^*(t),t) - v(x,t)) + c(l(x,t))\right\} + \frac{\partial v(x,t)}{\partial t}
\end{aligned} \tag{58}
$$

where x^* is the optimal return point that maximizes firm's profit.

Proposition 2. *Fix the time path of interest rate $r(t) > 0$, the curvature of the profit function $B > 0$, the volatility of shocks $\sigma > 0$, transitional inflation $\mu(t)$ and the thresholds $x^*(t)$. Moreover, we assume $-\infty < x(t) < \infty$ (i.e., bounds are infinite for the state). Consider a time-dependent function $\Lambda(\cdot,t) : x(t) \in (-\infty,\infty) \to \mathbb{R}_+$ that is differentiable and increasing on $(x^*(t),\infty)$, decreasing on $(-\infty, x^*(t))$ and with $\lim_{x\to\overline{x}}\Lambda(x,t) = \lim_{x\to\underline{x}}\Lambda(x,t)$. Let u be the unique solution of a partial differential equation*

$$
[r(t) + \Lambda(x,t)]u(x,t) - \frac{\partial u(x,t)}{\partial t} = 2Bx + \mu(t)\frac{\partial u(x,t)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2} \tag{59}
$$

for $x(t) \in [-\infty,\infty]$ and all t. Define $U(x,t) = \int_{x^*(t)}^{x} u(z,t)dz$ for all $x(t) \in [-\infty,\infty]$ and all t. There exist a cost function $c(\cdot)$ which is both increasing and convex that rationalizes Λ with a value function that solves the firm's problem (1) if and only if

$$
U(x(t)\to\infty,t) = U(x(t)\to-\infty,t), u(x,t) \leq 0
$$

for all $x \in (-\infty, x^*(t))$ and all t, and $u(x) \geq 0$ for all $x \in (x^*(t),\infty)$ and all t.

It is easy to verify that any generalized hazard function of the form

$$
\Lambda(x,t) = \kappa x^2 - f(t)x + g(t) \tag{60}
$$

is a legitimate and rationalized generalized hazard function that rationalizes the value function. Furthermore, it follows that the optimal reinjection point $x^*(t)$ must be equal to $x^*(t) = \frac{f(t)}{2\kappa}$, given the form of the generalized hazard function above and $f^2(t) - 4\kappa g(t) = 0$ for all $t \geq 0$ and $\kappa > 0$.

Note that the generalized hazard function of form above with transitional inflation $\mu(t)$ can be easily used to recover the generalized hazard function with steady-state inflation and zero inflation. For example, when $f(t)$ is constant f, the generalized hazard function given above becomes the generalized hazard function with steady-state inflation as

$$\Lambda(x) = \kappa x^2 - fx + g \tag{61}$$

When $f(t) = 0$, the generalized hazard function of (18) becomes the generalized hazard function with zero inflation as

$$\Lambda(x) = \kappa x^2 + g \tag{62}$$

The generalized hazard function $\Lambda(x,t) = \kappa x^2 - f(t)x + g(t)$ can be rewritten in a more compact way by the relation $f^2(t) - 4\kappa g(t) = 0$ as

$$\begin{aligned} \Lambda(x,t) &= \kappa x^2 - f(t)x + \frac{f^2(t)}{4\kappa} \\ &= \kappa \left[x - \frac{f(t)}{2\kappa} \right]^2 \end{aligned} \tag{63}$$

which is the form of the generalized hazard function used for the remainder of the paper.

Note that $u(x,t)$ in PDE is a transformed time-dependent value function $v(x,t)$ that solves the HJB equation. Due to the fact that any HJB equation is a backward equation (i.e., running backward in time), we must assign a terminal condition to the HJB. That is, we can assume the terminal condition of the solution $u(x,t)$ is given by

$$u(x,T) = u_T(x) \tag{64}$$

In general, $u_T(x)$ represents the stationary solution of the value function solving the HJB (2), that is, $u_T(x)$ solves

$$[r + \Lambda(x)]u_T(x) = 2Bx + \mu u_T'(x) + \frac{\sigma^2}{2} u_T''(x) \tag{65}$$

where $\Lambda(x) = \kappa x^2$.

The PDE for $u(x,t)$ can be rewritten in terms of $w(x,t)$ as

$$\frac{\partial w(x,t)}{\partial t} = -\frac{1}{2}\sigma^2 \frac{\partial^2 w(x,t)}{\partial x^2} + \left(r(t) + \Lambda(x,t) + \left[\frac{\mu'(t)}{\sigma^2} \right] x + \frac{1}{2} \left[\frac{\mu(t)}{\sigma} \right]^2 \right) w(x,t) - 2Bxe^{\frac{\mu(t)}{\sigma^2} x} \tag{66}$$

where $u(x,t) = e^{-\frac{\mu(t)}{\sigma^2}x} w(x,t)$.

After plugging in $\Lambda(x,t) = \kappa x^2 - f(t)x + \frac{f^2(t)}{4\kappa}$, we get

$$
\begin{aligned}
-\frac{\partial w(x,t)}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 w(x,t)}{\partial x^2} - \left(\kappa x^2 - \left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x + r(t) + \frac{f^2(t)}{4\kappa} + \frac{1}{2}\left[\frac{\mu(t)}{\sigma}\right]^2\right) w(x,t) \\
+ 2Bxe^{\frac{\mu(t)}{\sigma^2}x}
\end{aligned}
\tag{67}
$$

where $u(x,t) = e^{-\frac{\mu(t)}{\sigma^2}x} w(x,t)$ with terminal condition $u_T(x) = e^{-\frac{\mu(T)}{\sigma^2}x} w_T(x)$.

Now, in order to find the time path of distribution of price changes corresponding to the transitional inflation dynamics $-\mu(t)$, we must find a relationship between the density of price gaps x and the density of price changes $\Delta(p)$, because for most part of our analysis we use price gap x as the state variable of the firm, rather than price change $\Delta(p)$. In our case, we have an unbounded boundary of the state support. Therefore, based on Alvarez, Lippi, and Oskolkov (2022), the version of the relationship between time evolution of density of price changes, $q(\Delta p, t)$, and the time evolution of density of price gaps, $p(x,t)$, can be stated as

$$
q(\Delta p, t) = q(-x, t) = \frac{\Lambda(x,t)p(x,t)}{\int_{-\infty}^{\infty} \Lambda(x,t)p(x,t)dx} \tag{68}
$$

where the density of x is denoted by $p(x,t)$ which is the time evolution of the probability density function of price gaps, which is given by a time-dependent Kolmogorov Forward Equation (KFE) with the generalized hazard function $\Lambda(x,t) = \kappa x^2 - f(t)x + \frac{f^2(t)}{4\kappa}$ as

$$
\frac{\partial p(x,t)}{\partial t} = -\mu(t)\frac{\partial p(x,t)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 p(x,t)}{\partial x^2} - \left(\kappa x^2 - f(t)x + \frac{f^2(t)}{4\kappa}\right) p(x,t) \tag{69}
$$

for $x \neq x^*$ and with initial density distribution $p(x,0) = p_0(x)$. Note that the equation is a time-dependent KFE with time-dependent drift $\mu(t)$ and it can be transformed into an auxiliary time-dependent KFE without a drift term as

$$\frac{\partial\psi(x,t)}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2\psi(x,t)}{\partial x^2} - \left(\kappa x^2 - \left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x + \frac{f^2(t)}{4\kappa} + \frac{1}{2}\left[\frac{\mu(t)}{\sigma}\right]^2\right)\psi(x,t) \tag{70}$$

where $p(x,t) = e^{\frac{\mu(t)}{\sigma^2}x}\psi(x,t)$. Technically, it turns out $\psi(x,t)$ is easier to be solved than $p(x,t)$ analytically, although both of them are not easy to solve in a standard way due to the both time and state dependent coefficient in the equation. Once we have solved $\psi(x,t)$, then our ultimate goal of solving $p(x,t)$ can be easily reached by applying equation $p(x,t) = e^{\frac{\mu(t)}{\sigma^2}x}\psi(x,t)$.

3.2 Theoretical Framework for Path Integrals with Time-Varying Inflation $-\mu(t)$

We first establish the equivalence of path integral formulation with generalized hazard function in the context of transitional inflation $\Lambda^{\psi}(x,t) = \kappa x^2 - \left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x + \frac{f^2(t)}{4\kappa} + \frac{1}{2}\left[\frac{\mu(t)}{\sigma}\right]^2$ and the KFE given by equation (10), so that we can use the path integral formulation for solving KFE given by equation (10).

Proposition 3. *Path Integral formulation in its real version is equivalent to the KFE formulation.*

To get the path integral representation for $\psi(x,t)$ with a transitional inflation $\mu(t)$ and generalized the hazard function corresponding to $\psi(x,t)$, $\Lambda^{\psi}(x,t) = \kappa x^2 - \left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x + \frac{f^2(t)}{4\kappa} + \frac{1}{2}\left[\frac{\mu(t)}{\sigma}\right]^2$, we formulate the following integral for the kernel $K(x,t;y,0)$ which represents, in the context of $\psi(x,t)$, the transition probability of price gap going from y at time $t=0$ to x at time t as

$$K(x,t;y,0) = \int_y^x \exp\{-\frac{1}{\sigma^2}\int_0^t\left[\frac{1}{2}\dot{z}^2(\tau) + \sigma^2\Lambda^{\psi}(z,\tau)\right]d\tau\}\mathcal{D}z(\tau) \tag{71}$$

where z denotes the any possible transitional path between price gap y at time $t=0$ and x at time t. $\mathcal{D}$ explicitly refers to the fact that the integral is taken with respect to all the possible paths of z between y and x (i.e., the path integrals).

Next, we give two related propositions. First one is the kernel with time-varying inflation written in an standard algebraic form. The second proposition is the kernel with time-varying inflation written in terms of eigenvalues and eigenfunctions. The reason we come up with two versions of the kernel is because it turns out the first kernel is only useful to the density distribution, and the second kernel is useful for the survival function $Pr(\cdot)$ and the expectation component in the impulse response function.

Proposition 4. *The kernel $K^{\mu(t)}(x_b, t_b; x_a, t_a)$ with time-varying inflation $-\mu(t)$, generalized hazard function $\Lambda(x,t) = \kappa\Big[x - \frac{f(t)}{2\kappa}\Big]^2$ and volatility σ written in a standard algebraic form that will be applied to the calculation of density of x is given by*

$$\begin{aligned} K^{\mu(t)}(x_b, t_b; x_a, t_a) = & \left(\frac{\sqrt{2\kappa}\sigma}{2\pi\sigma^2 \sinh\sqrt{2\kappa}\sigma(t_b - t_a)}\right)^{1/2} \exp\left(-\frac{1}{2\sigma^2}\int_{t_a}^{t_b}\mu^2(t)dt\right) \\ & \times \exp\left(\frac{\mu(t)}{\sigma^2}x_b\right)\exp\left(-\frac{1}{4\kappa}\int_{t_a}^{t_b} f^2(t)dt\right)\exp\{-\frac{1}{\sigma^2}S_{cl}\} \end{aligned} \tag{72}$$

where

$$\begin{aligned} S_{cl} = & \frac{1}{2}\sigma\sqrt{2\kappa}\left[\frac{(x_a^2 + x_b^2)\cosh\sigma\sqrt{2\kappa}T - 2x_a x_b}{\sinh\sigma\sqrt{2\kappa}T}\right] \\ & - \frac{1}{2}\sigma\sqrt{2\kappa}\left[\frac{\Big(x_a - x_b\cosh\sigma\sqrt{2\kappa}T\Big)\Big(\frac{\sigma}{\sqrt{2\kappa}}\int_{t_a}^{t_b}\Big[f(s) - \frac{\mu'(s)}{\sigma^2}\Big]\sin\sigma\sqrt{2\kappa}(t_b - s)ds\Big)}{\sinh\sigma\sqrt{2\kappa}T}\right] \\ & - \frac{1}{2}\sigma^2 x_b\int_{t_a}^{t_b}\left[f(s) - \frac{\mu'(s)}{\sigma^2}\right]\cos\sigma\sqrt{2\kappa}(t_b - s)ds \\ & - \frac{\sigma^2\Big(x_b - x_a\exp\{-\sigma\sqrt{2\kappa}T\} + \frac{\sigma}{\sqrt{2\kappa}}\int_{t_a}^{t_b}\Big[f(s) - \frac{\mu'(s)}{\sigma^2}\Big]\sin\sigma\sqrt{2\kappa}(t_b - s)ds\Big)}{4\sinh\sigma\sqrt{2\kappa}T} \\ & \times \int_{t_a}^{t_b}\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]\exp\{\sigma\sqrt{2\kappa}(t - t_a)\}dt \\ & - \frac{\sigma^2\Big(x_a - (x_b + \frac{\sigma}{\sqrt{2\kappa}}\int_{t_a}^{t_b}\Big[f(s) - \frac{\mu'(s)}{\sigma^2}\Big]\sin\sigma\sqrt{2\kappa}(t_b - s)ds)\exp\{-\sigma\sqrt{2\kappa}T\}\Big)}{4\sinh\sigma\sqrt{2\kappa}T} \\ & \times \int_{t_a}^{t_b}\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]\exp\{\sigma\sqrt{2\kappa}(t_b - t)\}dt \\ & + \frac{1}{2}\frac{\sigma^3}{\sqrt{2\kappa}}\int_{t_a}^{t_b}\int_{t_a}^{t}\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]\left[f(s) - \frac{\mu'(s)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t - s)dsdt \end{aligned} \tag{73}$$

Proposition 5. *The kernel $\mathcal{K}^{\mu(t)}(x_b, t_b; x_a, t_a)$ with time-varying inflation $-\mu(t)$, gen-*

eralized hazard function $\Lambda(x,t) = \kappa\left[x - \frac{f(t)}{2\kappa}\right]^2$ *and volatility* σ *written in the form of eigenvalues and eigenfunctions that will be applied to the calculation of survival function and the expectation component of the impulse response function is given by*

$$\mathcal{K}^{\mu(t)}(x_b, t_b; x_a, t_a) = e^{-\frac{\mu(t)}{\sigma^2}x_b} \sum_m \sum_n G_{mn}(t_b, t_a)\phi_m(x_b)\phi_n(x_a)e^{-\frac{1}{\sigma^2}\lambda_m t_b} \tag{74}$$

where

$$\phi_m(x_b) = \frac{1}{(2^m m!)^{1/2}}\left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/4} H_m\left(\sqrt{\frac{\sqrt{\kappa}\sigma}{\sigma^2}}x_b\right)e^{-\left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)x_b^2}.$$

$$\phi_n(x_a) = \frac{1}{(2^n n!)^{1/2}}\left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/4} H_n\left(\sqrt{\frac{\sqrt{\kappa}\sigma}{\sigma^2}}x_a\right)e^{-\left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)x_a^2}$$

$$\lambda_m = \sqrt{\kappa}\sigma^3\left(m + \frac{1}{2}\right)$$

$$\begin{aligned} G_{mn}(t_b, t_a) &= \exp\left(-\frac{1}{2\sigma^2}\int_{t_a}^{t_b}\mu^2(t)dt\right)\exp\left(-\frac{1}{4\kappa}\int_{t_a}^{t_b}f^2(t)dt\right) \\ &\times \frac{G_{00}}{\sqrt{m!n!}}\sum_{r=0}^{k}\frac{m!}{(m-r)!r!}\frac{n!}{(n-r)!r!}r! \\ &\times \left(\frac{1}{\sqrt{2\sigma^2\sqrt{\kappa}\sigma}}\int_{t_a}^{t_b}\sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]e^{-\sqrt{\kappa}\sigma t}dt\right)^{n-r} \\ &\times \left(\frac{1}{\sqrt{2\sigma^2\sqrt{\kappa}\sigma}}\int_{t_a}^{t_b}\sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]e^{\sqrt{\kappa}\sigma t}dt\right)^{m-r} \end{aligned} \tag{75}$$

where $k = \min(m, n)$ *and*

$$G_{00} = \exp\left(-\frac{1}{2\sigma^2}\int_{t_a}^{t_b}\mu^2(t)dt\right)\exp\left(-\frac{1}{4\kappa}\int_{t_a}^{t_b}f^2(t)dt\right)$$

$$\times\exp\left(-\frac{1}{2\sigma^2\sqrt{\kappa}\sigma}\int_{t_a}^{t_b}\int_{t_a}^{t}\sigma^4\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]\left[f(s) - \frac{\mu'(s)}{\sigma^2}\right]e^{-\sqrt{\kappa}\sigma(t-s)}dsdt\right).$$

3.3 Applications

Next, we apply the two propositions provided above to perform two exercises. First, we study the impulse response of output to a monetary shock with time-varying inflation, sticky prices, and generalized hazard function. Second, we explore the impulse response of output to a monetary shock following an uncertainty shock with time-varying inflation, sticky prices, and generalized hazard function. Note that since the inflation considered in this section is no longer zero but time-varying, it follows that we will need to use the formula established in the introduction relating $H(t)$ and $G(t)$.

We first study impulse response of output to a monetary shock and later after this first exercise explore the impulse response of output to a monetary shock following an uncertainty shock. Recall that

$$H(t) = \sum_{n=1}^{\infty} Pr(t \in [\tau_{n-1}, \tau_n]) G(t \in [\tau_{n-1}, \tau_n])$$

where we define $\tau_0 = 0$, $H(t)$ is the impulse response function with firm's reinjection, $G(t)$ is the impulse response function of output without firm's reinjection, and the $Pr(\cdot)$ is the corresponding probability for each of these τs to be the stopping time at which the firm makes price adjustment and in literature the $Pr(\cdot)$ is called survival function. In particular, $G(t \in [\tau_{n-1}, \tau_n])$, by definition, is given by

$$G(t \in [\tau_{n-1}, \tau_n]) = \int_{-\infty}^{\infty} \mathbb{E}\big[-1_{\{\tau_{n-1} \leq t \leq \tau_n\}} x(t) | x(0) = x\big] \left[p^{\delta,t}(x, \tau_{n-1}) - p^{0,t}(x, \tau_{n-1})\right] dx \tag{76}$$

where $p^{\delta,t}(x, \tau_{n-1})$ is the density distribution of x at time τ_{n-1} right after their return to $x^*(t = \tau_{n-1}) = f(\tau_{n-1})/2\kappa$ and $p^{0,t}(x, \tau_{n-1})$ is the density distribution of x at time τ_{n-1} right before their return to $x^*(\tau_{n-1})$. We further note that $p^{\delta,t}(x, \tau_{n-1})$, except for $n = 1$ where the monetary shock happens, should be the Dirac delta function, that is, starting $n = 2$, $p^{\delta,t}(x, \tau_{n-1}) = \delta(x - f(\tau_{n-1})/2\kappa)$, and the $p^{0,t}(x, \tau_{n-1})$, except for $n = 1$ and $n = 2$, should be $\int_{-\infty}^{\infty} K^{\mu(t)}(x, \tau_{n-1}; y, \tau_{n-2}) \delta(y - f(\tau_{n-2})/2\kappa) dy = K^{\mu(t)}\Big(x, \tau_{n-1}; \frac{f(\tau_{n-2})}{2\kappa}, \tau_{n-2}\Big)$ for all $n \geq 2$. To summarize, we get

$$p^{\delta,t}(x, \tau_{n-1}) = \delta(x - f(\tau_{n-1})/2\kappa) \tag{77}$$

for all $n \geq 2$, and

$$p^{0,t}(x, \tau_{n-1}) = K^{\mu(t)}\left(x, \tau_{n-1}; \frac{f(\tau_{n-2})}{2\kappa}, \tau_{n-2}\right) \tag{78}$$

for all $n \geq 3$.

Second, the $Pr(t \in [\tau_{n-1}, \tau_n])$ is calculated by

$$Pr(t \in [\tau_{n-1}, \tau_n]) = \int_{-\infty}^{\infty} \mathcal{K}^{\mu(t)}\left(\frac{f(t)}{2\kappa}, t; y, \tau_{n-1}\right)dy - \int_{-\infty}^{\infty} \mathcal{K}^{\mu(t)}\left(\frac{f(t)}{2\kappa}, t; y, \tau_{n-2}\right)dy \tag{79}$$

for all $n \geq 2$. And

$$Pr(t \in [0 = \tau_0, \tau_1]) = \int_{-\infty}^{\infty} \mathcal{K}^{\mu(t)}\left(\frac{f(t)}{2\kappa}, t; y, \tau_0 = 0\right)dy \tag{80}$$

for $n = 1$.

Third, the expectation $\mathbb{E}\big[-1_{\{\tau_{n-1} \leq t \leq \tau_n\}} x(t) | x(0) = x\big]$ is calculated by

$$\begin{aligned} \mathbb{E}\big[-1_{\{\tau_{n-1} \leq t \leq \tau_n\}} x(t) | x(0) = x\big] = &- \int_{-\infty}^{\infty} \mathcal{K}^{\mu(t)}(x, t; y, \tau_{n-1}) y dy \\ &+ \int_{-\infty}^{\infty} \mathcal{K}^{\mu(t)}(x, t; y, \tau_{n-2}) y dy \end{aligned} \tag{81}$$

for all $n \geq 2$. And

$$\mathbb{E}\big[-1_{\{0 = \tau_0 \leq t \leq \tau_1\}} x(t) | x(0) = x\big] = - \int_{-\infty}^{\infty} \mathcal{K}^{\mu(t)}(x, t; y, \tau_0 = 0) y dy \tag{82}$$

for $n = 1$.

Last, for $n = 1$, the initial distribution of x at time $t = \tau_0 = 0$ before the monetary shock is given by

$$p^{0,t}(x, \tau_0 = 0) = \left(\frac{\sqrt{2\kappa}}{\pi\sigma}\right)^{1/2} e^{\left(-\frac{\sqrt{2\kappa}}{\sigma} x^2\right)} \tag{83}$$

and with the arrival of monetary shock of size δ at time $t = \tau_0 = 0$, the initial distribution of x at time $t = \tau_0 = 0$ after the monetary shock is given by

$$p^{\delta,t}(x, \tau_0 = 0) = \left(\frac{\sqrt{2\kappa}}{\pi\sigma}\right)^{1/2} e^{\left(-\frac{\sqrt{2\kappa}}{\sigma}(x-\delta)^2\right)} \tag{84}$$

Furthermore, for $n = 2$, the initial distribution of x at time $t = \tau_1$ before the reinjection is given by

$$p^{0,t}(x, \tau_1) = \int_{-\infty}^{\infty} K^{\mu(t)}(x, \tau_1; y, \tau_0 = 0)\left(\frac{\sqrt{2\kappa}}{\pi\sigma}\right)^{1/2} e^{\left(-\frac{\sqrt{2\kappa}}{\sigma}(y-\delta)^2\right)} dy \tag{85}$$

Therefore, the impulse response of output to a monetary shock of size δ with time-varying inflation $-\mu(t)$, sticky prices, generalized hazard function $\Lambda(x,t) = \kappa\left[x - \frac{f(t)}{2\kappa}\right]^2$, and volatility of the economy σ is expressed as

$$\begin{aligned} H(t) = & Pr(t \in [\tau_0 = 0, \tau_1])G(t \in [\tau_0 = 0, \tau_1]) \\ & + Pr(t \in [\tau_1, \tau_2])G(t \in [\tau_1, \tau_2]) \\ & + \sum_{n=3}^{\infty} Pr(t \in [\tau_{n-1}, \tau_n])G(t \in [\tau_{n-1}, \tau_n]) \end{aligned} \tag{86}$$

where

$$G(t \in [\tau_0 = 0, \tau_1]) = \int_{-\infty}^{\infty} \mathbb{E}\left[-1_{\{0=\tau_0 \le t \le \tau_1\}} x(t) | x(0) = x\right]\left[p^{\delta,t}(x, \tau_0 = 0) - p^{0,t}(x, \tau_0 = 0)\right] dx \tag{87}$$

$$G(t \in [\tau_1, \tau_2]) = \int_{-\infty}^{\infty} \mathbb{E}\left[-1_{\{\tau_1 \le t \le \tau_2\}} x(t) | x(0) = x\right]\left[p^{\delta,t}(x, \tau_1) - p^{0,t}(x, \tau_1)\right] dx \tag{88}$$

$$G(t \in [\tau_{n-1}, \tau_n]) = \int_{-\infty}^{\infty} \mathbb{E}\left[-1_{\{\tau_{n-1} \le t \le \tau_n\}} x(t) | x(0) = x\right]\left[p^{\delta,t}(x, \tau_{n-1}) - p^{0,t}(x, \tau_{n-1})\right] dx \tag{89}$$

for $n \geq 3$.

Next, we consider an exercise where there is a monetary shock following an uncertainty shock in the economy and the scenario is the same as in section 2 for the similar exercise with the only difference being the time-varying inflation, while section 2 is about zero inflation. So, we skip the description of the environment of the exercise and go directly into the derivation. In such a time-varying inflation environment, the result of the impulse response of output to a monetary shock following an uncertainty shock depends on when the monetary shock of size δ happens and which time interval $[\tau_{j-1}, \tau_j]$ for $j \in \{1, 2, ...n\}$ it belongs to. Assume the uncertainty shock occurs at time $t = 0$ and the monetary shock occurs s periods following the uncertainty shock, that is, the monetary shock occurs at time $t = s$ and s is located in between $[\tau_{j-1}, \tau_j]$ or equivalently, $s \in [\tau_{j-1}, \tau_j]$. Then, our question is that what is the effect of the uncertainty shock of size $\sigma - \sigma_0$ on the impulse response of output to the monetary shock of size δ? Specifically, $j = 1$ is in fact equivalent to any other case when $j \geq 2$ up to a change of the set of τs. Since we always can determine a different set of τs with different corresponding probabilistic occurrences by making $j = 1$, it follows that we only need to consider the case $j = 1$. Therefore, we only explore the case when $j = 1$ in this paper and all other cases $j \geq 2$ are equivalent to case $j = 1$ up to a change of the set of τs. Since there is an uncertainty shock at time $t = 0$, we assume the initial stationary distribution of the price gap x before uncertainty shock is given by

$$p^{0,0}(x) = \left(\frac{\sqrt{2\kappa}}{\pi\sigma_0}\right)^{1/2} e^{\left(-\frac{\sqrt{2\kappa}}{\sigma_0}x^2\right)} \tag{90}$$

where σ_0 is the initial steady-state volatility of the economy before the uncertainty shock.

At time $t = 0$, an uncertainty shock of size $\sigma - \sigma_0$ hits, so that the initial distribution of x after the uncertainty shock becomes

$$p^{\sigma-\sigma_0,0}(x) = \left(\frac{\sqrt{2\kappa}}{\pi\sigma}\right)^{1/2} e^{\left(-\frac{\sqrt{2\kappa}}{\sigma}x^2\right)} \tag{91}$$

where σ is the volatility of the economy after the uncertainty shock of size $\sigma - \sigma_0$. If $j = 1$, which implies that $\tau_0 = 0 \leq s \leq \tau_1$, then the time s distribution of price gap x before the monetary shock of size δ, $p^{0,t}(x, s)$, is given by

$$p^{0,t}(x, s) = \int_{-\infty}^{\infty} K^{\mu(t)}(x, s; y, 0)p^{\sigma-\sigma_0,0}(y)dy \tag{92}$$

and thus the time s distribution of price gap x after the monetary shock of size δ, $p^{\delta,t}(x, s)$, is given by

$$p^{\delta,t}(x, s) = \int_{-\infty}^{\infty} K^{\mu(t)}(x - \delta, s; y, 0)p^{\sigma-\sigma_0,0}(y)dy \tag{93}$$

Therefore, starting from time s until τ_1, that is, when $t \in [s, \tau_1]$, the impulse response function of output to the monetary shock is given by (note that in this time interval there is no firm's reinjection since τ_1^+ is the first time at which reinjection happens, where $+$ denotes the τ_1 belongs to the left-end point of the next consecutive time interval rather than the right-end point of the previous time interval)

$$G(t \in [s, \tau_1]) = \int_{-\infty}^{\infty} \mathbb{E}\big[-1_{\{s \le t \le \tau_1\}} x(t) | x(0) = x\big] \big[p^{\delta,t}(x, s) - p^{0,t}(x, s)\big] dx \tag{94}$$

where

$$\mathbb{E}\big[-1_{\{s \le t \le \tau_1\}} x(t) | x(0) = x\big] = -\int_{-\infty}^{\infty} \mathcal{K}^{\mu(t)}(x, t \in [s, \tau_1]; y, s) y dy \tag{95}$$

Furthermore, time τ_1^- distribution of price gap x is given by ($-$ denotes that the distribution is measured right before the reinjection at time τ_1)

$$p(x, \tau_1^-) = \int_{-\infty}^{\infty} K^{\mu(t)}(x, \tau_1; y, s) p^{\delta,t}(y, s) dy \tag{96}$$

Also, at time τ_1^+, reinjection happens, therefore the distribution becomes a Dirac function, that is,

$$p(x, \tau_1^+) = \delta\left(x - \frac{f(\tau_1)}{2\kappa}\right) \tag{97}$$

Hence, when $t \in [\tau_1, \tau_2]$, the impulse response becomes

$$G(t \in [\tau_1, \tau_2]) = \int_{-\infty}^{\infty} \mathbb{E}\big[-1_{\{\tau_1 \le t \le \tau_2\}} x(t) | x(0) = x\big] \big[p(x, \tau_1^+) - p(x, \tau_1^-)\big] dx \tag{98}$$

where

$$\mathbb{E}\big[-1_{\{\tau_1 \le t \le \tau_2\}} x(t) | x(0) = x\big] = -\int_{-\infty}^{\infty} \mathcal{K}^{\mu(t)}(x, t \in [\tau_1, \tau_2]; y, \tau_1) y dy \tag{99}$$

and all other components starting from $t \in [\tau_2, \tau_3]$ are the same as in the previous exercise when there is only monetary shock at time $t = 0$. That is, the impulse response of output to a monetary shock of size δ at time s following an uncertainty shock of size $\sigma - \sigma_0$ at time 0 with time-varying inflation is given by

$$\begin{aligned} H(t) = & Pr(t \in [s, \tau_1]) G(t \in [s, \tau_1]) \\ & + Pr(t \in [\tau_1, \tau_2]) G(t \in [\tau_1, \tau_2]) \\ & + \sum_{n=3}^{\infty} Pr(t \in [\tau_{n-1}, \tau_n]) G(t \in [\tau_{n-1}, \tau_n]) \end{aligned} \tag{100}$$

where

$$G(t \in [s, \tau_1]) = \int_{-\infty}^{\infty} \mathbb{E}\big[-1_{\{s \le t \le \tau_1\}} x(t) | x(0) = x\big] \big[p^{\delta,t}(x, s) - p^{0,t}(x, s)\big] dx$$

$$G(t \in [\tau_1, \tau_2]) = \int_{-\infty}^{\infty} \mathbb{E}\big[-1_{\{\tau_1 \le t \le \tau_2\}} x(t) | x(0) = x\big] \big[p(x, \tau_1^+) - p(x, \tau_1^-)\big] dx$$

$$G(t \in [\tau_{n-1}, \tau_n]) = \int_{-\infty}^{\infty} \mathbb{E}\big[-1_{\{\tau_{n-1} \le t \le \tau_n\}} x(t) | x(0) = x\big] \big[p^{\delta,t}(x, \tau_{n-1}) - p^{0,t}(x, \tau_{n-1})\big] dx$$

for $n \geq 3$ and all other elements involved in the expression are given above within the applications section. The aim of this section is to lay out the theoretical foundations and the framework for the firm's reinjection in a time-varying inflation environment of sticky prices. Since this is the first time in literature we can effectively handle such an important case, we view the theoretical foundation itself as groundbreaking even without any quantitative or any other practical considerations. Furthermore, all the theoretical foundations and the framework for the firm's reinjection in the context of sticky prices and time-varying inflation are not possible without resorting to the path integrals. From this perspective, we view path integrals as the ultimately fundamental knowledge for all of this.

4 Conclusion

This paper studies macroeconomic dynamics in a sticky price setting using path integrals. We analytically explored the transition dynamics of the economy with zero inflation in a sticky price setting with generalized hazard functions as well as the transition dynamics of the economy with time-varying inflation in the same sticky price setting. We solved the long-standing unsolved problem regarding the firm's reinjection which is indispensable part of the framework of sticky prices especially when the inflation is time-varying by expressing the impulse response function of output in such a setting. In a sticky price setting with generalized hazard functions with or without time-varying inflation, we explored the effect of monetary shock on the economy, the effect of the monetary shock following an uncertainty shock, and the effect of the monetary shock followed by an uncertainty shock, and showed that the order of the arrivals of the shocks also matters. The technique we used for our analysis, path integrals, turns out to be the most important tool that allows us to be able to explore so much things analytically that no one single previous work has ever done before. This paper aims to only help lay out the theoretical foundations for the topic and leave all other practical considerations like quantitative analysis or policy analysis to a future work. Even without any of such quantitative or empirical aspect of the analysis, we still view the work is groundbreaking and transformative in a way that the theoretical foundation itself alongside the introduction of path integrals into macroeconomics are important enough for this work to be an independent work.

A Appendix

A.1 Proof of Proposition 1

Proof. Define $L(\dot{x}, x, t) = \frac{1}{2}\dot{x}^2 + \sigma^2\Lambda(x)$, where σ^2 is the variance of the price gap x as a standard Brownian motion for the uncontrolled price process. Then,

$$L(\dot{x}, x, t) = \frac{1}{2}\dot{x}^2 + \kappa\sigma^2 x^2$$

and we have, by defining $S[x(t)] = \int_{t_a}^{t_b} L(\dot{x}, x, t)dt$,

$$S[x(t)] = S[\bar{x}(t) + y(t)]$$

that is,

$$\begin{aligned}
I &= S[x(t)] \\
&= \int_{t_a}^{t_b} \left(\frac{1}{2}\left(\dot{\bar{x}}^2 + 2\dot{\bar{x}}\dot{y} + \dot{y}^2\right) + \kappa\sigma^2\left(\bar{x}(t) + y(t)\right)^2\right)dt \\
&= \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t)\right)dt \\
&+ \int_{t_a}^{t_b} \left(\dot{\bar{x}}(t)\dot{y}(t) + \frac{1}{2}\dot{y}^2(t) + 2\kappa\sigma^2\bar{x}(t)y(t) + \kappa\sigma^2 y^2(t)\right)dt
\end{aligned}$$

Note that

$$\begin{aligned}
S_1 &= \int_{t_a}^{t_b} \big(\dot{\bar{x}}(t)\dot{y}(t) + 2\kappa\sigma^2\bar{x}(t)y(t)\big)dt \\
&= \int_{t_a}^{t_b} \dot{\bar{x}}(t)dy(t) + 2\kappa\sigma^2 \int_{t_a}^{t_b} \bar{x}(t)y(t)dt \\
&= [\dot{\bar{x}}(t)y(t)]_{t_a}^{t_b} - \int_{t_a}^{t_b} \ddot{\bar{x}}(t)y(t)dt + 2\kappa\sigma^2 \int_{t_a}^{t_b} \bar{x}(t)y(t)dt \\
&= -\int_{t_a}^{t_b} \big(2\kappa\sigma^2\bar{x}(t)\big)y(t)dt + 2\kappa\sigma^2 \int_{t_a}^{t_b} \bar{x}(t)y(t)dt \\
&= 0
\end{aligned}$$

where we have used $y(t_a) = y(t_b) = 0$ and from Euler Lagrange equation for $L(\dot{x}, x, t) = \frac{1}{2}\dot{x}^2 + \kappa\sigma^2 x^2$ to get $\ddot{\bar{x}}(t) = 2\kappa\sigma^2\bar{x}(t)$.

Therefore, we get

$$\begin{aligned}
S[x(t)] &= S[\bar{x}(t) + y(t)] \\
&= \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t)\right)dt + \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{y}^2(t) + \kappa\sigma^2 y^2(t)\right)dt \\
&= S[\bar{x}(t)] + S[y(t)]
\end{aligned}$$

where

$$S[\bar{x}(t)] = \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t)\right)dt$$

$$S[y(t)] = \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{y}^2(t) + \kappa\sigma^2 y^2(t)\right)dt$$

Therefore, we finally get

$$\begin{aligned}
K(b,a) &= \int_a^b \exp\left(-\frac{1}{\sigma^2}S[x(t)]\right)\mathcal{D}x(t) \\
&= \int_0^0 \exp\left(-\frac{1}{\sigma^2}S[\bar{x}(t) + y(t)]\right)\mathcal{D}y(t) \\
&= \int_0^0 \exp\left(-\frac{1}{\sigma^2}S[\bar{x}(t)] - \frac{1}{\sigma^2}S[y(t)]\right)\mathcal{D}y(t) \\
&= \exp\left(-\frac{1}{\sigma^2}S[\bar{x}(t)]\right) \int_0^0 \exp\left(-\frac{1}{\sigma^2}S[y(t)]\right)\mathcal{D}y(t)
\end{aligned}$$

That is, given the generalized hazard function $\Lambda(x) = \kappa x^2$, the corresponding kernel is given by

$$K(b,a) = \exp\left(-\frac{1}{\sigma^2}S[\bar{x}(t)]\right)\int_0^0 \exp\left(-\frac{1}{\sigma^2}S[y(t)]\right)\mathcal{D}y(t) \tag{101}$$

where

$$S[\bar{x}(t)] = \int_{t_a}^{t_b}\left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t)\right)dt$$

$$S[y(t)] = \int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t) + \kappa\sigma^2 y^2(t)\right)dt$$

First, we can compute $\int_0^0 \exp\left(-\frac{1}{\sigma^2}S[y(t)]\right)\mathcal{D}y(t)$ using the Fourier series method, and it turns out

$$\begin{aligned}\int_0^0 \exp\left(-\frac{1}{\sigma^2}S[y(t)]\right)\mathcal{D}y(t) &= \int_0^0 \exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t) + \kappa\sigma^2 y^2(t)\right)dt\right)\mathcal{D}y(t) \\ &= \left(\frac{\sqrt{2\kappa}\sigma}{2\pi\sigma^2 \sinh\sqrt{2\kappa}\sigma(t_b - t_a)}\right)^{1/2}\end{aligned}$$

Proof. To calculate $\int_0^0 \exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t) + \kappa\sigma^2 y^2(t)\right)dt\right)\mathcal{D}y(t)$, we first note that the path $y(t)$ has to meet the following requirement: $y(t_a = 0) = y(t_b = T) = 0$, and thus we can write $y(t)$ using Fourier series expansion as

$$y(t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi t}{T}\right) \tag{102}$$

Next, by direct plugging in and assuming that the time T is divided into discrete steps of length ϵ, our target of equation can be rewritten as

$$
\begin{aligned}
F(T) &= \int_0^0 \exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t)+\kappa\sigma^2 y^2(t)\right)dt\right)\mathcal{D}y(t) \\
&= J\frac{1}{A}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{-\frac{1}{2\sigma^2}\frac{T}{2}\sum_{n=1}^{N}\left[\left(\frac{n\pi}{T}\right)^2+2\kappa\sigma^2\right]a_n^2\} \\
&\times \frac{da_1}{A}\frac{da_2}{A}\cdots\frac{da_N}{A} \\
&= \frac{J}{A}\prod_{n=1}^{N}\int_{-\infty}^{\infty}\exp\{-\frac{1}{2\sigma^2}\frac{T}{2}\sum_{n=1}^{N}\left[\left(\frac{n\pi}{T}\right)^2+2\kappa\sigma^2\right]a_n^2\}\frac{da_n}{A} \\
&\propto \prod_{n=1}^{N}\left(\frac{n^2\pi^2}{T^2}+2\kappa\sigma^2\right)^{-1/2} \\
&= \prod_{n=1}^{N}\left(\frac{n^2\pi^2}{T^2}\right)^{-1/2}\prod_{n=1}^{N}\left(1+\frac{2\kappa\sigma^2T^2}{n^2\pi^2}\right)^{-1/2} \\
&\propto \left(\frac{\sinh\sqrt{2\kappa}\sigma T}{\sigma\sqrt{2\kappa}T}\right)^{-1/2}
\end{aligned} \tag{103}
$$

where we have applied Euler formula to the derivation from the second-to-last line to the last line.

$F(T)$ can be written in the form

$$
F(T) = C\left(\frac{\sinh\sqrt{2\kappa}\sigma T}{\sigma\sqrt{2\kappa}T}\right)^{-1/2} \tag{104}
$$

We consider the case in which $\sqrt{2\kappa}\sigma = 0$, since we already know from the previous derivations about the equivalence of path integral and KFE formulations that $F(T) = \left(\frac{1}{2\pi\sigma^2 T}\right)^{1/2}$ when $\sqrt{2\kappa}\sigma = 0$, which is just the inverse of the normalizing factor A. On the other hand, we also have (by utilizing L'Hopital's rule),

$$
\left(\frac{1}{2\pi\sigma^2 T}\right)^{1/2} = \lim_{\sqrt{2\kappa}\sigma\to 0}F(T) = \lim_{\sqrt{2\kappa}\sigma\to 0}C\left(\frac{\sinh\sqrt{2\kappa}\sigma T}{\sigma\sqrt{2\kappa}T}\right)^{-1/2} = C \tag{105}
$$

Therefore, our desired integral $F(T)$ is equal to

$$\begin{aligned} F(T) &= \left(\frac{1}{2\pi\sigma^2 T}\right)^{1/2}\left(\frac{\sinh\sqrt{2\kappa}\sigma T}{\sigma\sqrt{2\kappa}T}\right)^{-1/2} \\ &= \left(\frac{\sqrt{2\kappa}\sigma}{2\pi\sigma^2\sinh\sqrt{2\kappa}\sigma T}\right)^{1/2} \end{aligned} \tag{106}$$

where $T = t_b - t_a$. □

Hence, the kernel can be rewritten as

$$K(b,a) = \left(\frac{\sqrt{2\kappa}\sigma}{2\pi\sigma^2\sinh\sqrt{2\kappa}\sigma(t_b - t_a)}\right)^{1/2} \times \exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t)\right)dt\right) \tag{107}$$

Next, we compute

$$\exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t)\right)dt\right)$$

Note that since $\bar{x}$ can be any x due to that fact that $\bar{x}$ is just any arbitrary subset of x depending on our choice, it follows that $\bar{x}$ and x can be used interchangeably, i.e., $\bar{x} = x$. Therefore, computing

$$\exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t)\right)dt\right)$$

is equivalent to computing

$$\exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{x}^2(t) + \kappa\sigma^2 x^2(t)\right)dt\right)$$

From Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

associated with the $L = \frac{1}{2}\dot{x}^2(t) + \kappa\sigma^2 x^2(t)$ we get

$$\frac{d\dot{x}}{dt} - 2\kappa\sigma^2 x = 0,$$

or equivalently,

$$\ddot{x} = 2\kappa\sigma^2 x$$

which is an homogeneous linear second-order ODE whose solution can be written as

$$x(t) = A\exp\left\{\sigma\sqrt{2\kappa}(t-t_a)\right\} + B\exp\left\{\sigma\sqrt{2\kappa}(t_b-t)\right\} \tag{108}$$

and

$$\dot{x}(t) = A\sigma\sqrt{2\kappa}\exp\left\{\sigma\sqrt{2\kappa}(t-t_a)\right\} - B\sigma\sqrt{2\kappa}\exp\left\{\sigma\sqrt{2\kappa}(t_b-t)\right\} \tag{109}$$

Given the solution of $x(t)$ and $\dot{x}(t)$, we can proceed to compute

$$S_{cl} = \int_{t_a}^{t_b}\left(\frac{1}{2}\dot{x}^2(t) + \kappa\sigma^2 x^2(t)\right)dt$$

by simplification first and then direct substitution as follows.

$$\begin{aligned}
S_{cl} &= \int_{t_a}^{t_b}\left(\frac{1}{2}\dot{x}^2(t) + \kappa\sigma^2 x^2(t)\right)dt \\
&= \frac{1}{2}\int_{t_a}^{t_b}\dot{x}^2(t)dt + \int_{t_a}^{t_b}\kappa\sigma^2 x^2(t)dt \\
&= \frac{1}{2}\left([x\dot{x}]_{t_a}^{t_b} - \int_{t_a}^{t_b}x\ddot{x}dt\right) + \int_{t_a}^{t_b}\kappa\sigma^2 x^2(t)dt \\
&= \frac{1}{2}\left([x\dot{x}]_{t_a}^{t_b} - \int_{t_a}^{t_b}x\left(2\kappa\sigma^2 x\right)dt\right) + \int_{t_a}^{t_b}\kappa\sigma^2 x^2(t)dt \\
&= \frac{1}{2}[x(t)\dot{x}(t)]_{t_a}^{t_b}
\end{aligned} \tag{110}$$

where

$$x(t) = A\exp\left\{\sigma\sqrt{2\kappa}(t-t_a)\right\} + B\exp\left\{\sigma\sqrt{2\kappa}(t_b-t)\right\}$$

and

$$\dot{x}(t) = A\sigma\sqrt{2\kappa}\exp\left\{\sigma\sqrt{2\kappa}(t-t_a)\right\} - B\sigma\sqrt{2\kappa}\exp\left\{\sigma\sqrt{2\kappa}(t_b-t)\right\}$$

Hence, S_{cl} can be written as

$$S_{cl} = \frac{1}{2}\sigma\sqrt{2\kappa}\left[\frac{(x_a^2 + x_b^2)\cosh\sigma\sqrt{2\kappa}T - 2x_a x_b}{\sinh\sigma\sqrt{2\kappa}T}\right] \tag{111}$$

The kernel is thus calculated as

$$K(b,a) = \left(\frac{\sqrt{2\kappa}\sigma}{2\pi\sigma^2 \sinh \sqrt{2\kappa}\sigma(t_b - t_a)} \right)^{1/2} \exp\{-\frac{1}{\sigma^2} S_{cl}\} \tag{112}$$

□

A.2 Equivalence of KFE and Path Integrals with Zero Inflation

Proof. Let $\Lambda(x) = \kappa x^2$, then we have

$$p(x,t) = \int_{-\infty}^{\infty} K(x,t;y,0)p(y,0)dy \tag{113}$$

Note that for a short time interval ϵ, above equation can be rewritten as

$$p(x,t+\epsilon) = \frac{1}{A}\int_{-\infty}^{\infty} \exp\{-\frac{1}{\sigma^2}\epsilon L\left(\frac{x-y}{\epsilon}, \frac{x+y}{2}\right)\}p(y,t)dy \tag{114}$$

where $L = \frac{1}{2}\dot{x}^2 + \sigma^2\Lambda(x)$. And it can be rewritten as

$$\begin{aligned} p(x,t+\epsilon) = \frac{1}{A}\int_{-\infty}^{\infty} &\exp\{-\frac{1}{\sigma^2}\frac{(x-y)^2}{2\epsilon}\} \\ &\times \exp\{-\frac{1}{\sigma^2}\epsilon\sigma^2\Lambda\left(\frac{x+y}{2}, t\right)\}p(y,t)dy \end{aligned} \tag{115}$$

By a change of variables $y = x + \eta$, we have

$$\begin{aligned} p(x,t+\epsilon) = \frac{1}{A}\int_{-\infty}^{\infty} &\exp\{-\frac{\eta^2}{2\sigma^2\epsilon}\} \\ &\times \exp\{-\frac{1}{\sigma^2}\epsilon\sigma^2\Lambda\left(x+\frac{\eta}{2}, t\right)\}p(x+\eta,t)d\eta \end{aligned} \tag{116}$$

By expanding ψ in a power series, we get

$$\begin{aligned} p(x,t) + \epsilon\frac{\partial p}{\partial t} = \frac{1}{A}\int_{-\infty}^{\infty} &\exp\{-\frac{\eta^2}{2\sigma^2\epsilon}\} \\ &\times \left[1 - \frac{1}{\sigma^2}\epsilon\sigma^2\Lambda(x)\right]\left[p(x,t) + \eta\frac{\partial p}{\partial x} + \frac{\eta^2}{2}\frac{\partial^2 p}{\partial x^2}\right]d\eta \end{aligned} \tag{117}$$

Note that the leading term on the right-hand side is equal to (by Gaussian integral)

$$\frac{1}{A}\int_{-\infty}^{\infty} \exp\{-\frac{\eta^2}{2\sigma^2\epsilon}\}d\eta = \frac{1}{A}\left(2\pi\sigma^2\epsilon\right)^{1/2} \tag{118}$$

On the left-hand side, there is only $p(x,t)$, therefore, to let both sides agree to each other, A must be chosen so that $\frac{1}{A}(2\pi\sigma^2\epsilon)^{1/2} = 1$, that is,

$$A = \left(2\pi\sigma^2\epsilon\right)^{1/2} \tag{119}$$

Moreover, we can calculate the other two terms on the right-hand side of the expanded equation, that is,

$$\frac{1}{A}\int_{-\infty}^{\infty} \eta \exp\{-\frac{\eta^2}{2\sigma^2\epsilon}\}d\eta = 0 \tag{120}$$

$$\frac{1}{A}\int_{-\infty}^{\infty} \eta^2 \exp\{-\frac{\eta^2}{2\sigma^2\epsilon}\}d\eta = \sigma^2\epsilon \tag{121}$$

Finally, writing out the full version of the expanded equation using the fact that the second order of ϵ goes to zero, that is, $\epsilon^2 \to 0$, we get

$$p(x,t) + \epsilon\frac{\partial p}{\partial t} = p(x,t) - \frac{1}{\sigma^2}\epsilon\sigma^2\Lambda(x)p(x,t) + \frac{\sigma^2\epsilon}{2}\frac{\partial^2 p}{\partial x^2} \tag{122}$$

Simplifying it, we get

$$\frac{\partial p}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} - \Lambda(x)p(x,t) \tag{123}$$

where $\Lambda(x) = \kappa x^2$, which is the KFE equation. Hence, we have proven the equivalence of path integral formulation and the KFE formulation. $\square$

A.3 The Kernel Written in Eigenvalues and Eigenfunctions with Zero Inflation

Given the context, we define our corresponding Lagrangian $L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\kappa\sigma^2 x^2$, and define the classical action $S[x(t)] = \int_{t_a}^{t_b} L dt$, then it follows from above discussion about the path integrals that (i.e., let $x(t) = z(t)$ and $y(t) = \hat{z}(t)$)

$$S[x(t)] = S[\bar{x}(t) + y(t)] = \int_{t_a}^{t_b} \left(\frac{1}{2}\big(\dot{\bar{x}}^2 + 2\dot{\bar{x}}\dot{y} + \dot{y}^2\big) - \frac{1}{2}\kappa\sigma^2(\bar{x}(t) + y(t))^2 \right) dt$$

$$= \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) - \frac{1}{2}\kappa\sigma^2\bar{x}^2(t) \right) dt$$

$$+ \int_{t_a}^{t_b} \left(\dot{\bar{x}}(t)\dot{y}(t) + \frac{1}{2}\dot{y}^2(t) - \kappa\sigma^2\bar{x}(t)y(t) - \frac{1}{2}\kappa\sigma^2 y^2(t) \right) dt$$

Note that

$$\int_{t_a}^{t_b} \big(\dot{\bar{x}}(t)\dot{y}(t) - \kappa\sigma^2\bar{x}(t)y(t)\big) dt$$

$$= \int_{t_a}^{t_b} \dot{\bar{x}}(t)dy(t) - \kappa\sigma^2 \int_{t_a}^{t_b} \bar{x}(t)y(t)dt$$

$$= [\dot{\bar{x}}(t)y(t)]_{t_a}^{t_b} - \int_{t_a}^{t_b} \ddot{\bar{x}}(t)y(t)dt - \kappa\sigma^2 \int_{t_a}^{t_b} \bar{x}(t)y(t)dt$$

$$= -\int_{t_a}^{t_b} -\kappa\sigma^2\bar{x}(t)y(t)dt - \kappa\sigma^2 \int_{t_a}^{t_b} \bar{x}(t)y(t)dt = 0$$

where we have used $y(t_a) = y(t_b) = 0$ and from Euler Lagrange equation for $L = \frac{1}{2}\dot{\bar{x}}^2(t) - \frac{1}{2}\kappa\sigma^2\bar{x}^2(t)$, we get $\ddot{\bar{x}}(t) = -\kappa\sigma^2\bar{x}(t)$.

Therefore, we get

$$S[x(t)] = S[\bar{x}(t) + y(t)]$$

$$= \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) - \frac{1}{2}\kappa\sigma^2\bar{x}^2(t) \right) dt + \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{y}^2(t) - \frac{1}{2}\kappa\sigma^2 y^2(t) \right) dt$$

$$= S[\bar{x}(t)] + S[y(t)]$$

where

$$S[\bar{x}(t)] = \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) - \frac{1}{2}\kappa\sigma^2\bar{x}^2(t) \right) dt$$

$$S[y(t)] = \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{y}^2(t) - \frac{1}{2}\kappa\sigma^2 y^2(t) \right) dt$$

Therefore, by the definition of path integrals, the kernel $K(b, a)$ first can be expressed in terms of $x(t)$ so that the two end points of the path $x(t)$ are a and b, and second it can also be expressed in terms of the deviation of the actual path $x(t)$ from the arbitrarily fixed path $\bar{x}(t)$, $y(t)$, so that the two end points of $y(t)$ along the path are both zero. It follows from this line of logic that

$$K(b,a) = \int_a^b \exp\left(\frac{i}{\sigma^2} S[x(t)]\right) \mathcal{D}x(t)$$

$$= \int_0^0 \exp\left(\frac{i}{\sigma^2} S[\bar{x}(t) + y(t)]\right) \mathcal{D}y(t)$$

$$= \int_0^0 \exp\left(\frac{i}{\sigma^2} S[\bar{x}(t)] + \frac{i}{\sigma^2} S[y(t)]\right) \mathcal{D}y(t)$$

$$= \exp\left(\frac{i}{\sigma^2} S[\bar{x}(t)]\right) \int_0^0 \exp\left(\frac{i}{\sigma^2} S[y(t)]\right) \mathcal{D}y(t)$$

where i is the imaginary unit, i.e., $i^2 = -1$.

To summarize, given our economic setting where $\Lambda(x) = \kappa x^2$ with $\mu(t) = 0$ and volatility σ, and the implied Lagrangian $L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\kappa\sigma^2 x^2$, the implied kernel $K(b,a)$ is written as

$$K(b,a) = \exp\left(\frac{i}{\sigma^2} S[\bar{x}(t)]\right) \int_0^0 \exp\left(\frac{i}{\sigma^2} S[y(t)]\right) \mathcal{D}y(t),$$

where

$$S[\bar{x}(t)] = \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) - \frac{1}{2}\kappa\sigma^2 \bar{x}^2(t)\right) dt$$

$$S[y(t)] = \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{y}^2(t) - \frac{1}{2}\kappa\sigma^2 y^2(t)\right) dt$$

Notice that the kernel takes the form of the product of two functions, namely, $\exp\left(\frac{i}{\sigma} S[\bar{x}(t)]\right)$, and $\int_0^0 \exp\left(\frac{i}{\sigma} S[y(t)]\right) \mathcal{D}y(t)$. Later in this section, we will compute the first component, here we aim to compute the second component. That is, we aim to compute

$$\int_0^0 \exp\left(\frac{i}{\sigma^2} \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{y}^2(t) - \frac{1}{2}\kappa\sigma^2 y^2(t)\right) dt\right) \mathcal{D}y(t)$$

Since $y(t_a) = y(t_b) = 0$, it follows that, without loss of generality, $y(t)$ can be written as

$$y(t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi(t-t_a)}{t_b - t_a}$$

By the fact that

$$\int_{t_a}^{t_b} \cos \frac{n\pi(t-t_a)}{t_b-t_a} \cos \frac{m\pi(t-t_a)}{t_b-t_a} dt = 0, n \neq m$$

$$\int_{t_a}^{t_b} \cos \frac{n\pi(t-t_a)}{t_b-t_a} \cos \frac{m\pi(t-t_a)}{t_b-t_a} dt = \frac{1}{2}, n = m$$

we get

$$\frac{1}{2}\int_{t_a}^{t_b} \dot{y}^2(t)dt = \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{n\pi}{t_b-t_a}\frac{m\pi}{t_b-t_a} a_n a_m \int_{t_a}^{t_b} \cos \frac{n\pi(t-t_a)}{t_b-t_a} \cos \frac{m\pi(t-t_a)}{t_b-t_a} dt$$

$$= \frac{1}{2}\frac{t_b-t_a}{2}\sum_{n=1}^{\infty}\left(\frac{n\pi}{t_b-t_a}\right)^2 a_n^2,$$

and

$$\frac{1}{2}\kappa\sigma^2\int_{t_a}^{t_b} y^2(t)dt = \frac{1}{2}\omega^2\left(\frac{t_b-t_a}{2}\right)\sum_{n=1}^{\infty} a_n^2$$

Plugging in, we get

$$\int_0^0 \exp\left(\frac{i}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t) - \frac{1}{2}\kappa\sigma^2 y^2(t)\right)dt\right)\mathcal{D}y(t)$$

$$= \left(\frac{M}{A}\right)\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(\left(\frac{i}{2\sigma^2}\right)\left(\frac{t_b-t_a}{2}\right)\sum_{n=1}^{N}\left(\left(\frac{n\pi}{t_b-t_a}\right)^2 - \kappa\sigma^2\right)a_n^2\right)\times\frac{da_1}{A}\cdots\frac{da_N}{A}$$

$$= \left(\frac{M}{A}\right)\prod_{n=1}^{N}\int_{-\infty}^{\infty}\exp\left(\left(\frac{i}{2\sigma^2}\right)\left(\frac{t_b-t_a}{2}\right)\sum_{n=1}^{N}\left(\left(\frac{n\pi}{t_b-t_a}\right)^2 - \kappa\sigma^2\right)a_n^2\right)\frac{da_n}{A}$$

$$= \left(\frac{M}{A}\right)\prod_{n=1}^{N}\left(\frac{2}{\epsilon(t_b-t_a)}\right)^{1/2}\left(\frac{n^2\pi^2}{(t_b-t_a)^2} - \kappa\sigma^2\right)^{-1/2}$$

$$= \left(\frac{M}{A}\right)\left(\frac{2}{\epsilon(t_b-t_a)}\right)^{1/2}\prod_{n=1}^{N}\left(\frac{n^2\pi^2}{(t_b-t_a)^2} - \kappa\sigma^2\right)^{-1/2}$$

$$= \left(\frac{M}{A}\right)\left(\frac{2}{\epsilon(t_b - t_a)}\right)^{1/2} \prod_{n=1}^{N}\left(\frac{n^2\pi^2}{(t_b - t_a)^2}\right)^{-1/2} \prod_{n=1}^{N}\left(1 - \frac{\kappa\sigma^2(t_b - t_a)^2}{n^2\pi^2}\right)^{-1/2}$$

$$= \left(\frac{M}{A}\right)\left(\frac{2}{\epsilon(t_b - t_a)}\right)^{1/2} \prod_{n=1}^{N}\left(\frac{n^2\pi^2}{(t_b - t_a)^2}\right)^{-1/2} \lim_{N\to\infty} \prod_{n=1}^{N}\left(1 - \frac{\kappa\sigma^2(t_b - t_a)^2}{n^2\pi^2}\right)^{-1/2}$$

$$= \left(\frac{M}{A}\right)\left(\frac{2}{\epsilon(t_b - t_a)}\right)^{1/2} \prod_{n=1}^{N}\left(\frac{n^2\pi^2}{(t_b - t_a)^2}\right)^{-1/2} \left(\frac{\sin\sqrt{\kappa}\sigma(t_b - t_a)}{\sqrt{\kappa}\sigma(t_b - t_a)}\right)^{-1/2}$$

That is, we have found that

$$\int_0^0 \exp\left(\frac{i}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t) - \frac{1}{2}\kappa\sigma^2 y^2(t)\right)dt\right)\mathcal{D}y(t)$$

$$= \left(\frac{M}{A}\right)\left(\frac{2}{\epsilon(t_b - t_a)}\right)^{1/2} \prod_{n=1}^{N}\left(\frac{n^2\pi^2}{(t_b - t_a)^2}\right)^{-1/2} \left(\frac{\sin\sqrt{\kappa}\sigma(t_b - t_a)}{\sqrt{\kappa}\sigma(t_b - t_a)}\right)^{-1/2}$$

When $\kappa = 0$, it reduces to the case (the free particle case in physics) where we know the value of

$$\int_0^0 \exp\left(\frac{i}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t)\right)dt\right)\mathcal{D}y(t) = \left(\frac{1}{2\pi i\sigma^2(t_b - t_a)}\right)^{1/2}$$

Therefore, let $\kappa \to 0$ in

$$\int_0^0 \exp\left(\frac{i}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t) - \frac{1}{2}\kappa\sigma^2 y^2(t)\right)dt\right)\mathcal{D}y(t)$$

$$= \left(\frac{M}{A}\right)\left(\frac{2}{\epsilon(t_b - t_a)}\right)^{1/2} \prod_{n=1}^{N}\left(\frac{n^2\pi^2}{(t_b - t_a)^2}\right)^{-1/2} \left(\frac{\sin\sqrt{\kappa}\sigma(t_b - t_a)}{\sqrt{\kappa}\sigma(t_b - t_a)}\right)^{-1/2},$$

we get

$$\left(\frac{1}{2\pi i\sigma^2(t_b - t_a)}\right)^{1/2} = \left(\frac{M}{A}\right)\left(\frac{2}{\epsilon(t_b - t_a)}\right)^{1/2} \prod_{n=1}^{N}\left(\frac{n^2\pi^2}{(t_b - t_a)^2}\right)^{-1/2} \lim_{\kappa\to 0}\left(\frac{\sin\sqrt{\kappa}\sigma(t_b - t_a)}{\sqrt{\kappa}\sigma(t_b - t_a)}\right)^{-1/2}$$

By L'Hopital's rule, we have

$$\lim_{\kappa \to 0} \left(\frac{\sin \sqrt{\kappa}\sigma(t_b - t_a)}{\sqrt{\kappa}\sigma(t_b - t_a)} \right)^{-1/2} = 1$$

Therefore, we get

$$\left(\frac{M}{A}\right)\left(\frac{2}{\epsilon(t_b - t_a)}\right)^{1/2} \prod_{n=1}^{N} \left(\frac{n^2\pi^2}{(t_b - t_a)^2}\right)^{-1/2} = \left(\frac{1}{2\pi i\sigma^2(t_b - t_a)}\right)^{1/2}$$

Plugging back in

$$\int_0^0 \exp\left(\frac{i}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t) - \frac{1}{2}\kappa\sigma^2 y^2(t)\right)dt\right)\mathcal{D}y(t)$$

$$= \left(\frac{M}{A}\right)\left(\frac{2}{\epsilon(t_b - t_a)}\right)^{1/2} \prod_{n=1}^{N} \left(\frac{n^2\pi^2}{(t_b - t_a)^2}\right)^{-1/2} \left(\frac{\sin \sqrt{\kappa}\sigma(t_b - t_a)}{\sqrt{\kappa}\sigma(t_b - t_a)} \right)^{-1/2},$$

we get

$$\int_0^0 \exp\left(\frac{i}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t) - \frac{1}{2}\kappa\sigma^2 y^2(t)\right)dt\right)\mathcal{D}y(t) = \left(\frac{1}{2\pi i\sigma^2(t_b - t_a)}\right)^{1/2} \left(\frac{\sin \sqrt{\kappa}\sigma(t_b - t_a)}{\sqrt{\kappa}\sigma(t_b - t_a)} \right)^{-1/2}$$

Simplifying it, we get

$$\int_0^0 \exp\left(\frac{i}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t) - \frac{1}{2}\kappa\sigma^2 y^2(t)\right)dt\right)\mathcal{D}y(t) = \left(\frac{\sqrt{\kappa}\sigma}{2\pi i\sigma^2 \sin\sqrt{\kappa}\sigma(t_b - t_a)}\right)^{1/2}$$

Hence, the kernel $K(b,a)$ for the dynamics of the economy triggered by a monetary shock to the zero-inflation steady state with quadratic generalized hazard function $\Lambda(x) = \kappa x^2$ and volatility of the uncontrolled price process σ can thus be expressed by

$$K(b,a) = \left(\frac{\sqrt{\kappa}\sigma}{2\pi i\sigma^2 \sin\sqrt{\kappa}\sigma(t_b - t_a)}\right)^{1/2} \exp\left(\frac{i}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{\bar{x}}^2(t) - \frac{1}{2}\kappa\sigma^2 \bar{x}^2(t)\right)dt\right)$$

Now, we calculate $\exp\left(\frac{i}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{\bar{x}}^2(t) - \frac{1}{2}\kappa\sigma^2\bar{x}^2(t)\right)dt\right)$. From Euler-Lagrangian, we get $\ddot{x} = -\kappa\sigma^2 x$ and can solve for x as (define $T = t_b - t_a$)

$$x(t) = \frac{x_b - x_a \cos\sqrt{\kappa}\sigma T}{\sin\sqrt{\kappa}\sigma T} \sin\sqrt{\kappa}\sigma t + x_a \cos\sqrt{\kappa}\sigma t,$$

and thus

$$\dot{x}(t) = \sqrt{\kappa}\sigma \frac{x_b - x_a \cos\sqrt{\kappa}\sigma T}{\sin\sqrt{\kappa}\sigma T}\cos\sqrt{\kappa}\sigma t - \sqrt{\kappa}\sigma x_a \sin\sqrt{\kappa}\sigma t,$$

which implies

$$\dot{x}(T) = \sqrt{\kappa}\sigma \frac{x_b - x_a \cos\sqrt{\kappa}\sigma T}{\sin\sqrt{\kappa}\sigma T}\cos\sqrt{\kappa}\sigma T - \sqrt{\kappa}\sigma x_a \sin\sqrt{\kappa}\sigma T$$

$$\dot{x}(0) = \sqrt{\kappa}\sigma \frac{x_b - x_a \cos\sqrt{\kappa}\sigma T}{\sin\sqrt{\kappa}\sigma T}$$

Therefore,

$$S_{cl} = \frac{1}{2}\int_{t_a}^{t_b}\left(\dot{x}^2 - \kappa\sigma^2 x^2\right)dt$$

$$= \frac{1}{2}\left([\dot{x}x]_{t_a}^{t_b} - \int_{t_a}^{t_b} x\ddot{x}dt - \int_{t_a}^{t_b}\kappa\sigma^2 x^2 dt\right)$$

$$= \frac{1}{2}\left([\dot{x}x]_{t_a}^{t_b} - \int_{t_a}^{t_b} x\left(-\kappa\sigma^2 x\right)dt - \int_{t_a}^{t_b}\kappa\sigma^2 x^2 dt\right)$$

$$= \frac{1}{2}[\dot{x}x]_{t_a}^{t_b}$$

$$= \frac{1}{2}[\dot{x}x]_0^{T=t_b-t_a}$$

$$= \frac{1}{2}(x(T)\dot{x}(T) - x(0)\dot{x}(0))$$

$$= \frac{1}{2}(x_b\dot{x}(T) - x_a\dot{x}(0))$$

$$= \frac{1}{2}\left(x_b\left(\sqrt{\kappa}\sigma \frac{x_b - x_a \cos\sqrt{\kappa}\sigma T}{\sin\sqrt{\kappa}\sigma T}\cos\sqrt{\kappa}\sigma T - \sqrt{\kappa}\sigma x_a \sin\sqrt{\kappa}\sigma T\right) - x_a\left(\sqrt{\kappa}\sigma \frac{x_b - x_a \cos\sqrt{\kappa}\sigma T}{\sin\sqrt{\kappa}\sigma T}\right)\right)$$

$$= \frac{\sqrt{\kappa}\sigma}{2\sin\sqrt{\kappa}\sigma T}\left(\left(x_b^2 + x_a^2\right)\cos\sqrt{\kappa}\sigma T - 2x_b x_a\right)$$

The kernel for the dynamics of the economy triggered by a monetary shock to the zero-inflation steady state with quadratic generalized hazard function $\Lambda(x) = \kappa x^2$ and volatility of the uncontrolled price process σ can thus be finally expressed as

$$K(x_b, t_b; x_a, t_a)$$
$$= \left(\frac{\sqrt{\kappa}\sigma}{2\pi i \sigma^2 \sin\sqrt{\kappa}\sigma(t_b - t_a)}\right)^{1/2} \exp\left(\frac{i\sqrt{\kappa}\sigma}{2\sigma^2 \sin\sqrt{\kappa}\sigma(t_b - t_a)}\left((x_b^2 + x_a^2)\cos\sqrt{\kappa}\sigma(t_b - t_a) - 2x_b x_a\right)\right)$$

Next, we establish the relationship between the kernel $K(x_b, t_b; x_a, t_a)$ and the corresponding eigenfunctions $\psi_n(x)$ of the KFE, or equivalently the eigenfunctions $\psi_n(x)$ of its complex counterpart—the Schrodinger equation—in this economic setting (note that the KFE and its corresponding complex counterpart, the Schrodinger equation, share the same eigenfunctions $\psi_n(x)$ and eigenvalues λ_n. In physics the eigenvalues λ_n are called energy levels and eigenfunctions $\psi_n(x)$ are called eigenstates). In particular, we ask the following question: if $f(x)$ is the known density distribution of x at the time t_a, what is the density distribution of x at time t_b? In fact, the time-dependent solution to the complex version of KFE or the Schrodinger equation $p(x,t)$ at any time t can be written as

$$p(x,t) = \sum_{n=1}^{\infty} c_n e^{-(i/\sigma^2)\lambda_n t} \psi_n(x),$$

but at time t_a we have

$$f(x) = p(x, t_a) = \sum_{n=1}^{\infty} c_n e^{-(i/\sigma^2)\lambda_n t_a} \psi_n(x),$$

on the other hand, any $f(x)$ can be expressed as a linear combinations of the eigenfunctions that forms an orthogonal basis, that is, $f(x)$ can be generally written as

$$f(x) = \sum_{n=1}^{\infty} a_n \psi_n(x)$$

and the coefficient a_n can be written in terms of the eigenfunctions as

$$a_n = \int_{-\infty}^{\infty} \psi_n^*(x) f(x) dx$$

where $\psi_n^*(x)$ is the complex conjugate of $\psi_n(x)$.

Therefore, given all this, we have

$$f(x) = p(x,t_a) = \sum_{n=1}^{\infty} c_n e^{-(i/\sigma^2)\lambda_n t_a}\psi_n(x) = \sum_{n=1}^{\infty} a_n \psi_n(x)$$

from which we conclude that (due to the orthogonality of the eigenfunctions $\psi_n(x)$)

$$c_n = a_n e^{+(i/\sigma^2)\lambda_n t_a}$$

Putting this into $p(x,t) = \sum_{n=1}^{\infty} c_n e^{-(i/\sigma^2)\lambda_n t}\psi_n(x)$, we get at time t_b $p(x,t)$ can be written as

$$p(x,t_b) = \sum_{n=1}^{\infty} c_n e^{-(i/\sigma^2)\lambda_n t_b}\psi_n(x) = \sum_{n=1}^{\infty} a_n e^{-(i/\sigma^2)\lambda_n (t_b - t_a)}\psi_n(x)$$

Now using the expression of the a_n given above, we can rewrite $p(x,t_b)$ as

$$p(x,t_b) = \sum_{n=1}^{\infty} \psi_n(x) e^{-(i/\sigma^2)\lambda_n (t_b - t_a)} \int_{-\infty}^{\infty} \psi_n^*(y) f(y) dy$$

or equivalently,

$$p(x,t_b) = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \psi_n(x)\psi_n^*(y) e^{-(i/\sigma^2)\lambda_n (t_b - t_a)} f(y) dy$$

This final expression determines the time path of density distribution of x at time t_b completely in terms of $f(x)$. Note that by definition of the transition measure $K(x,t_b;y,t_a)$, the density distribution $p(x,t_b)$ can also be expressed using the transition measure $K(\cdot)$ as

$$p(x,t_b) = \int_{-\infty}^{\infty} K(x,t_b;y,t_a) f(y) dy,$$

then, comparing the previous two equations, we get

$$K(x_b,t_b;x_a,t_a) = \sum_{n=1}^{\infty} \psi_n(x_b)\psi_n^*(x_a) e^{-(i/\sigma^2)\lambda_n (t_b - t_a)}$$

for $t_b > t_a$.

Armed with this, we can go on to solve for the eigenvalues λ_n and eigenfunctions

$\psi_n(x)$ associated with either the KFE or the corresponding Schrodinger equation for the transition dynamics of the price gap x in an economy where the monetary shock hits the steady state of the economy with zero inflation (i.e., $\mu(t) = 0$) and quadratic generalized hazard function $\Lambda(x) = \frac{1}{2}\kappa x^2$.

Proposition 6. *Given the quadratic generalized hazard function $\Lambda(x) = \frac{1}{2}\kappa x^2$ and the standard deviation of the Brownian motion process σ for the uncontrolled sticky price gap x, the eigenvalues $-\lambda_n$ is given by*

$$-\lambda_n = -\sqrt{\kappa}\sigma^3\left(n + \frac{1}{2}\right)$$

and the eigenfunctions $\phi_n(x)$ is given by a Fredholm integral equation of the first kind

$$\int_{-\infty}^{\infty}\left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/4} e^{-(\sqrt{\kappa}\sigma/2\sigma^2)(x-a)^2}\phi_n(x)dx = \left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)^{\frac{n}{2}} \frac{a^n}{\sqrt{n!}} e^{-\frac{\sqrt{\kappa}\sigma}{4\sigma^2}a^2}$$

Proof. To show this, we assume the transition amplitude to go from any state $\psi(x)$ to another state $\chi(x)$ of the detrended sticky price process is denoted by $< \chi|1|\psi >$ which is defined by

$$< \chi|1|\psi > = \int\int \chi^*(x_b, t_b)K(b, a)\psi(x_a, t_a)dx_a dx_b$$

where $K(b, a)$ denotes the transition amplitude from state a to state b.

Suppose $\psi(x)$ and $\chi(x)$ are expanded in terms of the orthogonal functions $\phi_n(x)$, thus we get

$$\psi(x) = \sum_n \psi_n\phi_n(x)$$

$$\chi(x) = \sum_n \chi_n\phi_n(x)$$

It follows from the fact the kernel $K(x_b, t_b; x_a, t_a)$ can be written as $K(x_b, t_b; x_a, t_a) = \sum_{n=1}^{\infty}\phi_n(x_b)\phi_n^*(x_a)e^{-(i/\sigma^2)\lambda_n(t_b-t_a)}$ that the transition amplitude can be rewritten as

$$< \chi|1|\psi > = \int\int \chi^*(x_b, t_b)\sum_{n=1}^{\infty}\phi_n(x_b)\phi_n^*(x_a)e^{-(i/\sigma^2)\lambda_n(t_b-t_a)}\psi(x_a, t_a)dx_a dx_b$$

$$= \sum_n \int\int \chi^*(x_b, t_b)\phi_n(x_b)\phi_n^*(x_a)\psi(x_a, t_a)e^{-(i/\sigma^2)\lambda_n(t_b-t_a)}dx_a dx_b$$

$$= \sum_n \left(\int \chi^*(x_b, t_b)\phi_n(x_b)dx_b \right) \left(\int \psi(x_a, t_a)\phi_n^*(x_a)dx_a \right) e^{-(i/\sigma^2)\lambda_n T}$$

$$= \sum_n \chi_n^* \psi_n e^{-(i/\sigma^2)\lambda_n T}$$

where in the last line we have used $\chi_n^* = \int \chi^*(x)\phi_n(x)dx$ and $\psi_n = \int \psi(x)\phi_n^*(x)dx$ due to the orthogonal functions $\phi_n(x)$ and $T = t_b - t_a$.

Therefore, we get

$$\int\int \chi^*(x_b, t_b)K(b,a)\psi(x_a, t_a)dx_a dx_b = \sum_n \chi_n^* \psi_n e^{-(i/\sigma^2)\lambda_n T}$$

Next, suppose we choose a special pair of functions $\psi(x)$ and $\chi(x)$ for which the expansion on the right hand side of above equation is simple, then after obtaining the functions ψ_n we could get some information about functions $\phi_n(x)$. Suppose we choose the functions $\psi(x)$ and $\chi(x)$ as

$$\psi(x) = \left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2} \right)^{1/4} e^{-(\sqrt{\kappa}\sigma/2\sigma^2)(x-a)^2}$$

$$\chi(x) = \left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2} \right)^{1/4} e^{-(\sqrt{\kappa}\sigma/2\sigma^2)(x-b)^2}$$

These functions above represent Gaussian distributions centered about a and b, respectively. We therefore can set $\psi_n = \psi_n(a)$ and $\chi_n = \psi_n(b)$, then we get

$$\int\int \chi^*(x_b, t_b)K(b,a)\psi(x_a, t_a)dx_a dx_b = \sum_n \psi_n^*(b)\psi_n(a) e^{-(i/\Sigma^2)(\lambda_n - \omega_0)T}$$

We know from our previous discussion that the kernel for a quadratic generalized hazard function (in physics they call it harmonic oscillator) is given by

$$K(b,a) = \left(\frac{\sqrt{\kappa}\sigma}{2\pi i \sigma^2 \sin\sqrt{\kappa}\sigma T} \right)^{1/2} e^{\frac{i\sqrt{\kappa}\sigma}{2\sigma^2 \sin\sqrt{\kappa}\sigma T}\left(\left(x_b^2 + x_a^2\right)\cos\sqrt{\kappa}\sigma T - 2x_b x_a \right)}$$

Plugging in the left hand side of

$$\int\int \chi^*(x_b, t_b)K(b,a)\psi(x_a, t_a)dx_a dx_b = \sum_n \psi_n^*(b)\psi_n(a) e^{-(i/\sigma^2)\lambda_n T},$$

we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/4}e^{-(\sqrt{\kappa}\sigma/2\sigma^2)(x_b-b)^2}\left(\frac{\sqrt{\kappa}\sigma}{2\pi i\sigma^2\sin\sqrt{\kappa}\sigma T}\right)^{1/2}e^{\frac{i\sqrt{\kappa}\sigma}{2\sigma^2\sin\sqrt{\kappa}\sigma T}\left(\left(x_b^2+x_a^2\right)\cos\sqrt{\kappa}\sigma T-2x_bx_a\right)}$$

$$\times\left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/4}e^{-(\sqrt{\kappa}\sigma/2\sigma^2)(x_a-a)^2}dx_adx_b$$

$$=\left(\frac{\sqrt{\kappa}\sigma}{2\pi i\sigma^2\sin\sqrt{\kappa}\sigma T}\right)^{1/2}$$

$$\times\left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(\sqrt{\kappa}\sigma/2\sigma^2)(x_b-b)^2}e^{\frac{i\sqrt{\kappa}\sigma}{2\sigma^2\sin\sqrt{\kappa}\sigma T}\left(\left(x_b^2+x_a^2\right)\cos\sqrt{\kappa}\sigma T-2x_bx_a\right)}e^{-(\sqrt{\kappa}\sigma/2\sigma^2)(x_a-a)^2}dx_adx_b$$

Perform this double Gaussian integral which is somewhat lengthy but direct, we get the result of this integral as

$$\left(\frac{\sqrt{\kappa}\sigma}{2\pi i\sigma^2\sin\sqrt{\kappa}\sigma T}\right)^{1/2}\left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/2}$$

$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(\sqrt{\kappa}\sigma/2\sigma^2)(x_b-b)^2}e^{\frac{i\sqrt{\kappa}\sigma}{2\sigma^2\sin\sqrt{\kappa}\sigma T}\left(\left(x_b^2+x_a^2\right)\cos\sqrt{\kappa}\sigma T-2x_bx_a\right)}e^{-(\sqrt{\kappa}\sigma/2\sigma^2)(x_a-a)^2}dx_adx_b$$

$$=e^{-\frac{i\sqrt{\kappa}\sigma T}{2}-\frac{\sqrt{\kappa}\sigma}{4\sigma^2}\left(a^2+b^2-2abe^{-i\sqrt{\kappa}\sigma T}\right)}$$

Therefore, we get

$$e^{-\frac{i\sqrt{\kappa}\sigma T}{2}-\frac{\sqrt{\kappa}\sigma}{4\sigma^2}\left(a^2+b^2-2abe^{-i\sqrt{\kappa}\sigma T}\right)}=\sum_n\psi_n^*(b)\psi_n(a)e^{-(i/\sigma^2)\lambda_nT}$$

Or equivalently,

$$e^{-\frac{\sqrt{\kappa}\sigma}{4\sigma^2}\left(a^2+b^2\right)}e^{\frac{\sqrt{\kappa}\sigma ab}{2\sigma^2}e^{-i\sqrt{\kappa}\sigma T}}e^{-\frac{i\sqrt{\kappa}\sigma T}{2}}=\sum_n\psi_n^*(b)\psi_n(a)e^{-(i/\sigma^2)\lambda_nT}$$

Expanding $e^{\frac{\sqrt{\kappa}\sigma ab}{2\sigma^2}e^{-i\sqrt{\kappa}\sigma T}}$ in powers of $e^{-i\sqrt{\kappa}\sigma T}$ as

$$e^{\frac{\sqrt{\kappa}\sigma ab}{2\sigma^2}e^{-i\sqrt{\kappa}\sigma T}}=1+\left(\frac{\sqrt{\kappa}\sigma ab}{2\sigma^2}\right)e^{-i\sqrt{\kappa}\sigma T}+\frac{1}{2!}\left(\frac{\sqrt{\kappa}\sigma ab}{2\sigma^2}\right)^2e^{-2i\sqrt{\kappa}\sigma T}+\frac{1}{3!}\left(\frac{\sqrt{\kappa}\sigma ab}{2\sigma^2}\right)^3e^{-3i\sqrt{\kappa}\sigma T}+...$$

Plugging in $e^{-\frac{\sqrt{\kappa}\sigma}{4\sigma^2}\left(a^2+b^2\right)} e^{\frac{\sqrt{\kappa}\sigma ab}{2\sigma^2} e^{-i\sqrt{\kappa}\sigma T}} e^{-\frac{i\sqrt{\kappa}\sigma T}{2}} = \sum_n \psi_n^*(b)\psi_n(a)e^{-(i/\sigma^2)\lambda_n T}$, we get

$$e^{-\frac{\sqrt{\kappa}\sigma}{4\sigma^2}\left(a^2+b^2\right)}\left(1+\left(\frac{\sqrt{\kappa}\sigma ab}{2\sigma^2}\right)e^{-i\sqrt{\kappa}\sigma T}+\frac{1}{2!}\left(\frac{\sqrt{\kappa}\sigma ab}{2\sigma^2}\right)^2 e^{-2i\sqrt{\kappa}\sigma T}+...\right)e^{-\frac{i\sqrt{\kappa}\sigma T}{2}}$$

$$= \sum_n \psi_n^*(b)\psi_n(a)e^{-(i/\sigma^2)\lambda_n T}$$

Or equivalently,

$$e^{-\frac{\sqrt{\kappa}\sigma}{4\sigma^2}\left(a^2+b^2\right)}\left(e^{-\frac{i\sqrt{\kappa}\sigma T}{2}}+\left(\frac{\sqrt{\kappa}\sigma ab}{2\sigma^2}\right)e^{-\left(1+\frac{1}{2}\right)i\sqrt{\kappa}\sigma T}+\frac{1}{2!}\left(\frac{\sqrt{\kappa}\sigma ab}{2\sigma^2}\right)^2 e^{-\left(2+\frac{1}{2}\right)i\sqrt{\kappa}\sigma T}+...\right)$$

$$= \sum_n \psi_n^*(b)\psi_n(a)e^{-(i/\sigma^2)\lambda_n T}$$

Comparing terms on both sides, we can solve

$$-\lambda_n = -\sqrt{\kappa}\sigma\sigma^2\left(n+\frac{1}{2}\right)$$

$$\psi_n(a) = \left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)^{\frac{n}{2}}\frac{a^n}{\sqrt{n!}}e^{-\frac{\sqrt{\kappa}\sigma}{4\sigma^2}a^2}$$

Since $\psi(x) = \sum_n \psi_n\phi_n(x)$, where $\phi_n(x)$ are orthogonal functions, it follows that

$$\int_{-\infty}^{\infty}\left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/4} e^{-(\sqrt{\kappa}\sigma/2\sigma^2)(x-a)^2}\phi_n(x)dx = \left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)^{\frac{n}{2}}\frac{a^n}{\sqrt{n!}}e^{-\frac{\sqrt{\kappa}\sigma}{4\sigma^2}a^2},$$

which is a type of Fredholm integral equation about $\phi_n(x)$ of the first kind that can be solved analytically. $\square$

Proposition 7. *Given above Proposition , that is, given eigenvalues*

$$-\lambda_n = -\sqrt{\kappa}\sigma\sigma^2\left(n+\frac{1}{2}\right)$$

and the eigenfunctions $\phi_n(x)$ is given by a Fredholm integral equation of the first kind

$$\int_{-\infty}^{\infty}\left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/4} e^{-(\sqrt{\kappa}\sigma/2\sigma^2)(x-a)^2}\phi_n(x)dx = \left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)^{\frac{n}{2}}\frac{a^n}{\sqrt{n!}}e^{-\frac{\sqrt{\kappa}\sigma}{4\sigma^2}a^2},$$

the eigenvalues $-\lambda_n$ *and eigenfunctions* $\phi_n(x)$ *can be finally written as*

$$-\lambda_n = -\sqrt{\kappa}\sigma^3\left(n+\frac{1}{2}\right)$$

$$\phi_n(x) = \frac{1}{(2^n n!)^{1/2}}\left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/4} H_n\left(\sqrt{\frac{\sqrt{\kappa}\sigma}{\sigma^2}}x\right)e^{-\left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)x^2}$$

where H *is the (physicist's) Hermite polynomial of degree* n *given by* $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$ *and* $n = 0, 1, 2, ...$

Let us rewrite the above integral equation in a form of convolution for the left hand side as

$$\int_{-\infty}^{\infty}\left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/4} e^{-(\sqrt{\kappa}\sigma/2\sigma^2)(a-x)^2}\phi_n(x)dx = \left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)^{\frac{n}{2}}\frac{a^n}{\sqrt{n!}}e^{-\frac{\sqrt{\kappa}\sigma}{4\sigma^2}a^2}$$

Clearly, if let

$$f(a-x) = \left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/4} e^{-(\sqrt{\kappa}\sigma/2\sigma^2)(a-x)^2}$$

$$g(a) = \left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)^{\frac{n}{2}}\frac{a^n}{\sqrt{n!}}e^{-\frac{\sqrt{\kappa}\sigma}{4\sigma^2}a^2},$$

then, the integral equation takes the form of

$$\int_{-\infty}^{\infty} f(a-x)\phi_n(x)dx = g(a),$$

and the left hand side is actually a convolution of $f(x)$ and $\phi_n(x)$. Thus, by taking Fourier transform to both sides of the equation just above, we get

$$\hat{f}(\xi)\hat{\phi}_n(\xi) = \hat{g}(\xi),$$

and therefore,

$$\hat{\phi}_n(\xi) = \frac{\hat{g}(\xi)}{\hat{f}(\xi)},$$

where

$$\hat{f}(\xi) = \int_{-\infty}^{\infty}\left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/4} e^{-(\sqrt{\kappa}\sigma/2\sigma^2)x^2}e^{-i\xi x}dx$$

$$\hat{g}(\xi) = \int_{-\infty}^{\infty}\left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)^{\frac{n}{2}}\frac{x^n}{\sqrt{n!}}e^{-\frac{\sqrt{\kappa}\sigma}{4\sigma^2}x^2}e^{-i\xi x}dx$$

That is,

$$\hat{\phi}_n(\xi) = \frac{\int_{-\infty}^{\infty} \left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)^{\frac{n}{2}} \frac{x^n}{\sqrt{n!}} e^{-\frac{\sqrt{\kappa}\sigma}{4\sigma^2}x^2} e^{-i\xi x} dx}{\int_{-\infty}^{\infty} \left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/4} e^{-(\sqrt{\kappa}\sigma/2\sigma^2)x^2} e^{-i\xi x} dx}$$

With some algebra and simplification, and then taking inverse Fourier transform to $\hat{\phi}_n(\xi)$, we get

$$\phi_n(x) = \frac{1}{\pi\sqrt{2}} \frac{i^n \left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)^{\frac{n}{2}} / \sqrt{n!}}{\left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/4}} \int_{-\infty}^{\infty} \frac{d^n\left(e^{-\frac{\sigma^2\xi^2}{\sqrt{\kappa}\sigma}}\right)}{d\xi^n} e^{\frac{\sigma^2\xi^2}{2\sqrt{\kappa}\sigma} + ix\xi} d\xi$$

Let us first rewrite $\frac{d^n\left(e^{-\frac{\sigma^2\xi^2}{\sqrt{\kappa}\sigma}}\right)}{d\xi^n} e^{\frac{\sigma^2\xi^2}{2\sqrt{\kappa}\sigma}}$ in terms of Hermite polynomials $H_n(\sqrt{\frac{\sigma^2}{\sqrt{\kappa}\sigma}}\xi)$ by the definition of $H_n(\sqrt{\frac{\sigma^2}{\sqrt{\kappa}\sigma}}\xi) = (-1)^n \frac{d^n\left(e^{-\left(\sqrt{\frac{\sigma^2}{\sqrt{\kappa}\sigma}}\xi\right)^2}\right)}{d\left(\sqrt{\frac{\sigma^2}{\sqrt{\kappa}\sigma}}\xi\right)^n} e^{\left(\sqrt{\frac{\sigma^2}{\sqrt{\kappa}\sigma}}\xi\right)^2}$, which leads to

$$\frac{d^n\left(e^{-\frac{\sigma^2\xi^2}{\sqrt{\kappa}\sigma}}\right)}{d\xi^n} e^{\frac{\sigma^2\xi^2}{2\sqrt{\kappa}\sigma}} = \frac{(-1)^n e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa}\sigma}}\xi\right)^2} H_n(\sqrt{\frac{\sigma^2}{\sqrt{\kappa}\sigma}}\xi)}{\left(\sqrt{\frac{\sqrt{\kappa}\sigma}{\sigma^2}}\right)^n}$$

Substituting in the expression for $\phi_n(x)$, we get

$$\phi_n(x) = \frac{1}{\pi\sqrt{2}} \frac{i^n \left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)^{\frac{n}{2}} / \sqrt{n!}}{\left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/4}} \int_{-\infty}^{\infty} \frac{(-1)^n e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa}\sigma}}\xi\right)^2} H_n(\sqrt{\frac{\sigma^2}{\sqrt{\kappa}\sigma}}\xi)}{\left(\sqrt{\frac{\sqrt{\kappa}\sigma}{\sigma^2}}\right)^n} e^{ix\xi} d\xi$$

Or equivalently,

$$\phi_n(x) = \frac{1}{\pi\sqrt{2}} \frac{i^n \left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)^{\frac{n}{2}} / \sqrt{n!}}{\left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/4}} \frac{(-1)^n}{\left(\sqrt{\frac{\sqrt{\kappa}\sigma}{\sigma^2}}\right)^n} \int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa}\sigma}}\xi\right)^2} H_n\left(\sqrt{\frac{\sigma^2}{\sqrt{\kappa}\sigma}}\xi\right) e^{ix\xi} d\xi$$

On the other hand, by generating function of Hermite Polynomial of $H_n\left(\sqrt{\frac{\sigma^2}{\sqrt{\kappa\sigma}}}\xi\right)$:

$$e^{2\sqrt{\frac{\sigma^2}{\sqrt{\kappa\sigma}}}\xi t-t^2}=\sum_{n=0}^{\infty}H_n\left(\sqrt{\frac{\sigma^2}{\sqrt{\kappa\sigma}}}\xi\right)\frac{t^n}{n!},$$

which can be multiplied by $e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa\sigma}}}\xi\right)^2}$ on both sides to get

$$e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa\sigma}}}\xi\right)^2+2\sqrt{\frac{\sigma^2}{\sqrt{\kappa\sigma}}}\xi t-t^2}=\sum_{n=0}^{\infty}e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa\sigma}}}\xi\right)^2}H_n\left(\sqrt{\frac{\sigma^2}{\sqrt{\kappa\sigma}}}\xi\right)\frac{t^n}{n!}$$

Taking inverse Fourier transform of the left hand side, we get

$$\begin{aligned}
&\mathcal{F}^{-1}\left(e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa\sigma}}}\xi\right)^2+2\sqrt{\frac{\sigma^2}{\sqrt{\kappa\sigma}}}\xi t-t^2}\right)(x)\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa\sigma}}}\xi\right)^2+2\sqrt{\frac{\sigma^2}{\sqrt{\kappa\sigma}}}\xi t-t^2}e^{ix\xi}d\xi\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa\sigma}}}\xi\right)^2+2\sqrt{\frac{\sigma^2}{\sqrt{\kappa\sigma}}}\xi t-t^2+ix\xi}d\xi\\
&=\frac{1}{2\pi}\sqrt{\frac{2\sqrt{\kappa\sigma}\pi}{\sigma^2}}e^{-\frac{\sqrt{\kappa\sigma}}{2\sigma^2}x^2}e^{2\sqrt{\frac{\sqrt{\kappa\sigma}}{\sigma^2}}xit+t^2}\\
&=\frac{1}{2\pi}\sqrt{\frac{2\sqrt{\kappa\sigma}\pi}{\sigma^2}}e^{-\frac{\sqrt{\kappa\sigma}}{2\sigma^2}x^2}\sum_{n=0}^{\infty}H_n\left(\sqrt{\frac{\sqrt{\kappa\sigma}}{\sigma^2}}x\right)\frac{(it)^n}{n!}
\end{aligned}$$

That is,

$$\mathcal{F}^{-1}\left(e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa\sigma}}}\xi\right)^2+2\sqrt{\frac{\sigma^2}{\sqrt{\kappa\sigma}}}\xi t-t^2}\right)(x)=\frac{1}{2\pi}\sqrt{\frac{2\sqrt{\kappa\sigma}\pi}{\sigma^2}}e^{-\frac{\sqrt{\kappa\sigma}}{2\sigma^2}x^2}\sum_{n=0}^{\infty}H_n\left(\sqrt{\frac{\sqrt{\kappa\sigma}}{\sigma^2}}x\right)\frac{(it)^n}{n!}$$

The inverse Fourier transform of the right hand side of the same equation is

$$\begin{aligned}
&\mathcal{F}^{-1}\left(\sum_{n=0}^{\infty}e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa\sigma}}}\xi\right)^2}H_n\left(\sqrt{\frac{\sigma^2}{\sqrt{\kappa\sigma}}}\xi\right)\frac{t^n}{n!}\right)(x)\\
&=\sum_{n=0}^{\infty}\mathcal{F}^{-1}\left(e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa\sigma}}}\xi\right)^2}H_n\left(\sqrt{\frac{\sigma^2}{\sqrt{\kappa\sigma}}}\xi\right)\frac{t^n}{n!}\right)(x)
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{F}^{-1} \left(e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa}\sigma}}\xi\right)^2} H_n\left(\sqrt{\frac{\sigma^2}{\sqrt{\kappa}\sigma}}\xi\right)\right)(x)$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa}\sigma}}\xi\right)^2} H_n\left(\sqrt{\frac{\sigma^2}{\sqrt{\kappa}\sigma}}\xi\right) e^{ix\xi} d\xi$$

That is,

$$\mathcal{F}^{-1}\left(\sum_{n=0}^{\infty} e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa}\sigma}}\xi\right)^2} H_n\left(\sqrt{\frac{\sigma^2}{\sqrt{\kappa}\sigma}}\xi\right)\frac{t^n}{n!}\right)(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa}\sigma}}\xi\right)^2} H_n\left(\sqrt{\frac{\sigma^2}{\sqrt{\kappa}\sigma}}\xi\right) e^{ix\xi} d\xi$$

Since the two inverse Fourier transforms are taken respect to the two sides of the one equation respectively, it follows that

$$\frac{1}{2\pi}\sqrt{\frac{2\sqrt{\kappa}\sigma\pi}{\sigma^2}} e^{-\frac{\sqrt{\kappa}\sigma}{2\sigma^2}x^2} \sum_{n=0}^{\infty} H_n\left(\sqrt{\frac{\sqrt{\kappa}\sigma}{\sigma^2}}x\right)\frac{(it)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa}\sigma}}\xi\right)^2} H_n\left(\sqrt{\frac{\sigma^2}{\sqrt{\kappa}\sigma}}\xi\right) e^{ix\xi} d\xi,$$

which leads to

$$\frac{t^n}{n!} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa}\sigma}}\xi\right)^2} H_n\left(\sqrt{\frac{\sigma^2}{\sqrt{\kappa}\sigma}}\xi\right) e^{ix\xi} d\xi = \frac{1}{2\pi}\sqrt{\frac{2\sqrt{\kappa}\sigma\pi}{\sigma^2}} e^{-\frac{\sqrt{\kappa}\sigma}{2\sigma^2}x^2} H_n\left(\sqrt{\frac{\sqrt{\kappa}\sigma}{\sigma^2}}x\right)\frac{(it)^n}{n!}$$

Or equivalently,

$$\int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa}\sigma}}\xi\right)^2} H_n\left(\sqrt{\frac{\sigma^2}{\sqrt{\kappa}\sigma}}\xi\right) e^{ix\xi} d\xi = \frac{2\pi n!}{t^n} \frac{1}{2\pi}\sqrt{\frac{2\sqrt{\kappa}\sigma\pi}{\sigma^2}} e^{-\frac{\sqrt{\kappa}\sigma}{2\sigma^2}x^2} H_n\left(\sqrt{\frac{\sqrt{\kappa}\sigma}{\sigma^2}}x\right)\frac{(it)^n}{n!}$$

Note that

$$\phi_n(x) = \frac{1}{\pi\sqrt{2}} \frac{i^n\left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)^{\frac{n}{2}}/\sqrt{n!}}{\left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/4}} \frac{(-1)^n}{\left(\sqrt{\frac{\sqrt{\kappa}\sigma}{\sigma^2}}\right)^n} \int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{\sigma^2}{2\sqrt{\kappa}\sigma}}\xi\right)^2} H_n\left(\sqrt{\frac{\sigma^2}{\sqrt{\kappa}\sigma}}\xi\right) e^{ix\xi} d\xi$$

By a direct substitution, we get

$$\phi_n(x) = \frac{1}{\pi\sqrt{2}} \frac{i^n\left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)^{\frac{n}{2}}/\sqrt{n!}}{\left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/4}} \frac{(-1)^n}{\left(\sqrt{\frac{\sqrt{\kappa}\sigma}{\sigma^2}}\right)^n} \frac{2\pi n!}{t^n} \frac{1}{2\pi}\sqrt{\frac{2\sqrt{\kappa}\sigma\pi}{\sigma^2}} e^{-\frac{\sqrt{\kappa}\sigma}{2\sigma^2}x^2} H_n\left(\sqrt{\frac{\sqrt{\kappa}\sigma}{\sigma^2}}x\right)\frac{(it)^n}{n!},$$

which can be further simplified as

$$\phi_n(x) = \frac{1}{(2^n n!)^{1/2}} \left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2} \right)^{1/4} H_n\left(\sqrt{\frac{\sqrt{\kappa}\sigma}{\sigma^2}} x \right) e^{-\left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)x^2}$$

Thus, the kernel can be written in terms of eigenvalues and eigenfunctions by utilizing the following property which is also given above as

$$K(x_b, t_b; x_a, t_a) = \sum_{n=1}^{\infty} \phi_n(x_b)\phi_n^*(x_a) e^{-(i/\sigma^2)\lambda_n(t_b - t_a)},$$

where

$$\lambda_n = \sqrt{\kappa}\sigma^3 \left(n + \frac{1}{2} \right)$$

$$\phi_n(x_b) = \frac{1}{(2^n n!)^{1/2}} \left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2} \right)^{1/4} H_n\left(\sqrt{\frac{\sqrt{\kappa}\sigma}{\sigma^2}} x_b \right) e^{-\left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)x_b^2}$$

$$\phi_n^*(x_a) = \frac{1}{(2^n n!)^{1/2}} \left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2} \right)^{1/4} H_n\left(\sqrt{\frac{\sqrt{\kappa}\sigma}{\sigma^2}} x_a \right) e^{-\left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)x_a^2}$$

A.4 Proof of Proposition 4

Proof. Define $L(\dot{x}, x, t) = \frac{1}{2}\dot{x}^2 + \sigma^2\Lambda(x, t)$, where σ^2 is the variance of the price gap x as a standard Brownian motion for the uncontrolled price process. Then,

$$L(\dot{x}, x, t) = \frac{1}{2}\dot{x}^2 + \kappa\sigma^2 x^2 - \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] x + \frac{\sigma^2 f^2(t)}{4\kappa} + \frac{1}{2}\mu^2(t)$$

and we have, by defining $S[x(t)] = \int_{t_a}^{t_b} L(\dot{x}, x, t)dt$,

$$S[x(t)] = S[\bar{x}(t) + y(t)]$$

that is,

$$
\begin{aligned}
I &= S[x(t)] \\
&= \int_{t_a}^{t_b} \left(\frac{1}{2}\left(\dot{\bar{x}}^2 + 2\dot{\bar{x}}\dot{y} + \dot{y}^2\right) + \kappa\sigma^2(\bar{x}(t) + y(t))^2 - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right](\bar{x}(t) + y(t)) + \frac{\sigma^2 f^2(t)}{4\kappa} + \frac{\mu^2(t)}{2} \right) dt \\
&= \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t) - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]\bar{x}(t) \right) dt \\
&+ \int_{t_a}^{t_b} \left(\dot{\bar{x}}(t)\dot{y}(t) + \frac{1}{2}\dot{y}^2(t) + 2\kappa\sigma^2\bar{x}(t)y(t) + \kappa\sigma^2 y^2(t) - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]y(t) \right) dt \\
&+ \frac{\sigma^2}{4\kappa}\int_{t_a}^{t_b} f^2(t)dt + \frac{1}{2}\int_{t_a}^{t_b} \mu^2(t)dt
\end{aligned}
$$

Note that

$$
\begin{aligned}
S_1 &= \int_{t_a}^{t_b} \left(\dot{\bar{x}}(t)\dot{y}(t) + 2\kappa\sigma^2\bar{x}(t)y(t) - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]y(t) \right) dt \\
&= \int_{t_a}^{t_b} \dot{\bar{x}}(t)dy(t) + 2\kappa\sigma^2\int_{t_a}^{t_b} \bar{x}(t)y(t)dt - \int_{t_a}^{t_b} \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]y(t)dt \\
&= [\dot{\bar{x}}(t)y(t)]_{t_a}^{t_b} - \int_{t_a}^{t_b} \ddot{\bar{x}}(t)y(t)dt + 2\kappa\sigma^2\int_{t_a}^{t_b} \bar{x}(t)y(t)dt - \int_{t_a}^{t_b} \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]y(t)dt \\
&= -\int_{t_a}^{t_b} \left(2\kappa\sigma^2\bar{x}(t) - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right] \right) y(t)dt + 2\kappa\sigma^2\int_{t_a}^{t_b} \bar{x}(t)y(t)dt - \int_{t_a}^{t_b} \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]y(t)dt \\
&= 0
\end{aligned}
$$

where we have used $y(t_a) = y(t_b) = 0$ and from Euler Lagrange equation for $L(\dot{x}, x, t) = \frac{1}{2}\dot{x}^2 + \kappa\sigma^2 x^2 - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x + \frac{\sigma^2 f^2(t)}{4\kappa} + \frac{1}{2}\mu^2(t)$ to get $\ddot{\bar{x}}(t) = 2\kappa\sigma^2\bar{x}(t) - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]$.

Therefore, we get

$$
\begin{aligned}
S[x(t)] &= S[\bar{x}(t) + y(t)] \\
&= \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t) - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]\bar{x}(t) \right) dt + \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{y}^2(t) + \kappa\sigma^2 y^2(t) \right) dt \\
&+ \frac{\sigma^2}{4\kappa}\int_{t_a}^{t_b} f^2(t)dt + \frac{1}{2}\int_{t_a}^{t_b} \mu^2(t)dt \\
&= S[\bar{x}(t)] + S[y(t)] + \frac{\sigma^2}{4\kappa}\int_{t_a}^{t_b} f^2(t)dt + \frac{1}{2}\int_{t_a}^{t_b} \mu^2(t)dt
\end{aligned}
$$

where

$$
S[\bar{x}(t)] = \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t) - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]\bar{x}(t) \right) dt
$$

$$S[y(t)] = \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{y}^2(t) + \kappa\sigma^2 y^2(t) \right) dt$$

Therefore, we finally get

$$\begin{aligned}
K(b,a) &= \int_a^b \exp\left(-\frac{1}{\sigma^2} S[x(t)] \right) \mathcal{D}x(t) \\
&= \int_0^0 \exp\left(-\frac{1}{\sigma^2} S[\bar{x}(t) + y(t)] - \frac{1}{4\kappa}\int_{t_a}^{t_b} f^2(t)dt - \frac{1}{2\sigma^2}\int_{t_a}^{t_b} \mu^2(t)dt \right) \mathcal{D}y(t) \\
&= \int_0^0 \exp\left(-\frac{1}{\sigma^2} S[\bar{x}(t)] - \frac{1}{\sigma^2} S[y(t)] - \frac{1}{4\kappa}\int_{t_a}^{t_b} f^2(t)dt - \frac{1}{2\sigma^2}\int_{t_a}^{t_b} \mu^2(t)dt \right) \mathcal{D}y(t) \\
&= \exp\left(-\frac{1}{4\kappa}\int_{t_a}^{t_b} f^2(t)dt \right) \exp\left(-\frac{1}{2\sigma^2}\int_{t_a}^{t_b} \mu^2(t)dt \right) \\
&\times \exp\left(-\frac{1}{\sigma^2} S[\bar{x}(t)] \right) \int_0^0 \exp\left(-\frac{1}{\sigma^2} S[y(t)] \right) \mathcal{D}y(t)
\end{aligned}$$

That is, given the generalized hazard function with transitional inflation, the corresponding kernel is given by

$$\begin{aligned}
K(b,a) = \exp\left(-\frac{1}{2\sigma^2}\int_{t_a}^{t_b} \mu^2(t)dt \right) \exp\left(-\frac{1}{4\kappa}\int_{t_a}^{t_b} f^2(t)dt \right) \\
\times \exp\left(-\frac{1}{\sigma^2} S[\bar{x}(t)] \right) \int_0^0 \exp\left(-\frac{1}{\sigma^2} S[y(t)] \right) \mathcal{D}y(t)
\end{aligned}$$

where

$$S[\bar{x}(t)] = \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t) - \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \bar{x}(t) \right) dt$$

$$S[y(t)] = \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{y}^2(t) + \kappa\sigma^2 y^2(t) \right) dt$$

First, we can compute $\int_0^0 \exp\left(-\frac{1}{\sigma^2} S[y(t)]\right) \mathcal{D}y(t)$ using the Fourier series method, and it turns out

$$
\begin{aligned}
\int_0^0 \exp\left(-\frac{1}{\sigma^2}S[y(t)]\right)\mathcal{D}y(t) &= \int_0^0 \exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t)+\kappa\sigma^2 y^2(t)\right)dt\right)\mathcal{D}y(t) \\
&= \left(\frac{\sqrt{2\kappa}\sigma}{2\pi\sigma^2\sinh\sqrt{2\kappa}\sigma(t_b-t_a)}\right)^{1/2}
\end{aligned}
$$

Proof. To calculate $\int_0^0 \exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t)+\kappa\sigma^2 y^2(t)\right)dt\right)\mathcal{D}y(t)$, we first note that the path $y(t)$ has to meet the following requirement: $y(t_a=0)=y(t_b=T)=0$, and thus we can write $y(t)$ using Fourier series expansion as

$$
y(t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi t}{T}\right) \tag{124}
$$

Next, by direct plugging in and assuming that the time T is divided into discrete steps of length ϵ, our target of equation can be rewritten as

$$
\begin{aligned}
F(T) &= \int_0^0 \exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{y}^2(t)+\kappa\sigma^2 y^2(t)\right)dt\right)\mathcal{D}y(t) \\
&= J\frac{1}{A}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{-\frac{1}{2\sigma^2}\frac{T}{2}\sum_{n=1}^{N}\left[\left(\frac{n\pi}{T}\right)^2+2\kappa\sigma^2\right]a_n^2\} \\
&\times \frac{da_1}{A}\frac{da_2}{A}\cdots\frac{da_N}{A} \\
&= \frac{J}{A}\prod_{n=1}^{N}\int_{-\infty}^{\infty}\exp\{-\frac{1}{2\sigma^2}\frac{T}{2}\sum_{n=1}^{N}\left[\left(\frac{n\pi}{T}\right)^2+2\kappa\sigma^2\right]a_n^2\}\frac{da_n}{A} \\
&\propto \prod_{n=1}^{N}\left(\frac{n^2\pi^2}{T^2}+2\kappa\sigma^2\right)^{-1/2} \\
&= \prod_{n=1}^{N}\left(\frac{n^2\pi^2}{T^2}\right)^{-1/2}\prod_{n=1}^{N}\left(1+\frac{2\kappa\sigma^2T^2}{n^2\pi^2}\right)^{-1/2} \\
&\propto \left(\frac{\sinh\sqrt{2\kappa}\sigma T}{\sigma\sqrt{2\kappa}T}\right)^{-1/2}
\end{aligned} \tag{125}
$$

where we have applied Euler formula to the derivation from the second-to-last line to the last line.

$F(T)$ can be written in the form

$$F(T) = C\left(\frac{\sinh\sqrt{2\kappa}\sigma T}{\sigma\sqrt{2\kappa}T}\right)^{-1/2} \tag{126}$$

We consider the case in which $\sqrt{2\kappa}\sigma = 0$, since we already know from the previous derivations about the equivalence of path integral and KFE formulations that $F(T) = \left(\frac{1}{2\pi\sigma^2 T}\right)^{1/2}$ when $\sqrt{2\kappa}\sigma = 0$, which is just the inverse of the normalizing factor A. On the other hand, we also have (by utilizing L'Hopital's rule),

$$\left(\frac{1}{2\pi\sigma^2 T}\right)^{1/2} = \lim_{\sqrt{2\kappa}\sigma\to 0} F(T) = \lim_{\sqrt{2\kappa}\sigma\to 0} C\left(\frac{\sinh\sqrt{2\kappa}\sigma T}{\sigma\sqrt{2\kappa}T}\right)^{-1/2} = C \tag{127}$$

Therefore, our desired integral $F(T)$ is equal to

$$\begin{aligned} F(T) &= \left(\frac{1}{2\pi\sigma^2 T}\right)^{1/2}\left(\frac{\sinh\sqrt{2\kappa}\sigma T}{\sigma\sqrt{2\kappa}T}\right)^{-1/2} \\ &= \left(\frac{\sqrt{2\kappa}\sigma}{2\pi\sigma^2\sinh\sqrt{2\kappa}\sigma T}\right)^{1/2} \end{aligned} \tag{128}$$

where $T = t_b - t_a$. $\square$

Hence, the kernel can be rewritten as

$$\begin{aligned} K(b,a) = &\left(\frac{\sqrt{2\kappa}\sigma}{2\pi\sigma^2\sinh\sqrt{2\kappa}\sigma(t_b - t_a)}\right)^{1/2} \exp\left(-\frac{1}{2\sigma^2}\int_{t_a}^{t_b}\mu^2(t)dt\right)\exp\left(-\frac{1}{4\kappa}\int_{t_a}^{t_b} f^2(t)dt\right) \\ &\times \exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t) - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]\bar{x}(t)\right)dt\right) \end{aligned}$$

Next, we compute

$$\exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t) - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]\bar{x}(t)\right)dt\right)$$

Note that since $\bar{x}$ can be any x due to that fact that $\bar{x}$ is just any arbitrary subset of x depending on our choice, it follows that $\bar{x}$ and x can be used interchangeably, i.e., $\bar{x} = x$. Therefore, computing

$$\exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{\bar{x}}^2(t) + \kappa\sigma^2\bar{x}^2(t) - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]\bar{x}(t)\right)dt\right)$$

is equivalent to computing

$$\exp\left(-\frac{1}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{x}^2(t) + \kappa\sigma^2 x^2(t) - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x(t)\right)dt\right)$$

From Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

associated with the $L = \frac{1}{2}\dot{x}^2(t) + \kappa\sigma^2 x^2(t) - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x(t) + \frac{\sigma^2 f^2(t)}{4\kappa} + \frac{1}{2}\mu^2(t)$ we get

$$\frac{d\dot{x}}{dt} - 2\kappa\sigma^2 x + \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right] = 0,$$

or equivalently,

$$\ddot{x} = 2\kappa\sigma^2 x - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]$$

which is an inhomogeneous linear second-order ODE whose solution can be written as

$$\begin{aligned} x(t) &= A\exp\{\sigma\sqrt{2\kappa}(t - t_a)\} + B\exp\{\sigma\sqrt{2\kappa}(t_b - t)\} \\ &\quad - \frac{1}{\sigma\sqrt{2\kappa}}\int_{t_a}^{t}\sigma^2\left[f(s) - \frac{\mu'(s)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t - s)ds \end{aligned} \tag{129}$$

and

$$\begin{aligned} \dot{x}(t) &= A\sigma\sqrt{2\kappa}\exp\{\sigma\sqrt{2\kappa}(t - t_a)\} - B\sigma\sqrt{2\kappa}\exp\{\sigma\sqrt{2\kappa}(t_b - t)\} \\ &\quad - \int_{t_a}^{t}\sigma^2\left[f(s) - \frac{\mu'(s)}{\sigma^2}\right]\cos\sigma\sqrt{2\kappa}(t - s)ds \end{aligned} \tag{130}$$

Given the solution of $x(t)$ and $\dot{x}(t)$, we can proceed to compute

$$S_{cl} = \int_{t_a}^{t_b}\left(\frac{1}{2}\dot{x}^2(t) + \kappa\sigma^2 x^2(t) - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x(t)\right)dt$$

by simplification first and then direct substitution as follows.

$$
\begin{aligned}
S_{cl} &= \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{x}^2(t) + \kappa\sigma^2 x^2(t) - \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] x(t) \right) dt \\
&= \frac{1}{2} \int_{t_a}^{t_b} \dot{x}^2(t) dt + \int_{t_a}^{t_b} \kappa\sigma^2 x^2(t) dt - \int_{t_a}^{t_b} \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] x(t) dt \\
&= \frac{1}{2} \left([x\dot{x}]_{t_a}^{t_b} - \int_{t_a}^{t_b} x\ddot{x} dt \right) + \int_{t_a}^{t_b} \kappa\sigma^2 x^2(t) dt - \int_{t_a}^{t_b} \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] x(t) dt \\
&= \frac{1}{2} \left([x\dot{x}]_{t_a}^{t_b} - \int_{t_a}^{t_b} x \left(2\kappa\sigma^2 x - \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \right) dt \right) + \int_{t_a}^{t_b} \kappa\sigma^2 x^2(t) dt \\
&- \int_{t_a}^{t_b} \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] x(t) dt \\
&= \frac{1}{2} [x(t)\dot{x}(t)]_{t_a}^{t_b} - \frac{1}{2} \int_{t_a}^{t_b} \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] x(t) dt
\end{aligned} \tag{131}
$$

where

$$
\begin{aligned}
x(t) = A \exp\{\sigma\sqrt{2\kappa}(t - t_a)\} + B \exp\{\sigma\sqrt{2\kappa}(t_b - t)\} \\
- \frac{1}{\sigma\sqrt{2\kappa}} \int_{t_a}^{t} \sigma^2 \left[f(s) - \frac{\mu'(s)}{\sigma^2} \right] \sin\sigma\sqrt{2\kappa}(t - s) ds
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{x}(t) = A\sigma\sqrt{2\kappa} \exp\{\sigma\sqrt{2\kappa}(t - t_a)\} - B\sigma\sqrt{2\kappa} \exp\{\sigma\sqrt{2\kappa}(t_b - t)\} \\
- \int_{t_a}^{t} \sigma^2 \left[f(s) - \frac{\mu'(s)}{\sigma^2} \right] \cos\sigma\sqrt{2\kappa}(t - s) ds
\end{aligned}
$$

We define some notations first, let

$$
X_S = \frac{1}{\sigma\sqrt{2\kappa}} \int_{t_a}^{t_b} \sigma^2 \left[f(s) - \frac{\mu'(s)}{\sigma^2} \right] \sin\sigma\sqrt{2\kappa}(t_b - s) ds \tag{132}
$$

$$
X_C = \int_{t_a}^{t_b} \sigma^2 \left[f(s) - \frac{\mu'(s)}{\sigma^2} \right] \cos\sigma\sqrt{2\kappa}(t_b - s) ds \tag{133}
$$

and the initial and terminal values are given by

$$
x_a = A + B \exp\{\sigma\sqrt{2\kappa}(t_b - t_a)\} \tag{134}
$$

$$
x_b = A \exp\{\sigma\sqrt{2\kappa}(t_b - t_a)\} + B - X_S \tag{135}
$$

$$
\dot{x}(t_a) = A\sigma\sqrt{2\kappa} - B\sigma\sqrt{2\kappa} \exp\{\sigma\sqrt{2\kappa}(t_b - t_a)\} \tag{136}
$$

$$\dot{x}(t_b) = A\sigma\sqrt{2\kappa}\exp\{\sigma\sqrt{2\kappa}(t_b - t_a)\} - B\sigma\sqrt{2\kappa} - X_C \tag{137}$$

Therefore, we can solve for A and B in terms of x_a and x_b as

$$A = \frac{x_b - x_a\exp\{-\sigma\sqrt{2\kappa}T\} + X_S}{\exp\{\sigma\sqrt{2\kappa}T\} - \exp\{-\sigma\sqrt{2\kappa}T\}} \tag{138}$$

$$B = \frac{x_a\exp\{\sigma\sqrt{2\kappa}T\} - x_b - X_S}{\exp\{2\sigma\sqrt{2\kappa}T\} - 1} \tag{139}$$

where $T = t_b - t_a$.

Then, we have

$$\begin{aligned} S_{cl} &= \frac{1}{2}[x(t)\dot{x}(t)]_{t_a}^{t_b} - \frac{1}{2}\int_{t_a}^{t_b}\sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x(t)dt \\ &= \frac{1}{2}\left[x_b\dot{x}(t_b) - x_a\dot{x}(t_a) - \sigma^2\int_{t_a}^{t_b}\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x(t)dt\right] \end{aligned} \tag{140}$$

First, let us calculate $x_b\dot{x}(t_b) - x_a\dot{x}(t_a)$ by a following direct substitution and calculation:

$$\begin{aligned} x_b\dot{x}(t_b) - x_a\dot{x}(t_a) &= x_b\Big[A\sigma\sqrt{2\kappa}\exp\{\sigma\sqrt{2\kappa}(t_b - t_a)\} - B\sigma\sqrt{2\kappa} - X_C\Big] \\ &- x_a\Big[A\sigma\sqrt{2\kappa} - B\sigma\sqrt{2\kappa}\exp\{\sigma\sqrt{2\kappa}(t_b - t_a)\}\Big] \\ &= x_b\Big[A\sigma\sqrt{2\kappa}\exp\{\sigma\sqrt{2\kappa}T\} - B\sigma\sqrt{2\kappa} - X_C\Big] \\ &- x_a\Big[A\sigma\sqrt{2\kappa} - B\sigma\sqrt{2\kappa}\exp\{\sigma\sqrt{2\kappa}T\}\Big] \\ &= A\sigma\sqrt{2\kappa}\Big(x_b\exp\{\sigma\sqrt{2\kappa}T\} - x_a\Big) + B\sigma\sqrt{2\kappa}\Big(x_a\exp\{\sigma\sqrt{2\kappa}T\} - x_b\Big) - x_bX_C \end{aligned} \tag{141}$$

where

$$A\sigma\sqrt{2\kappa} = \left(\frac{x_b - x_a\exp\{-\sigma\sqrt{2\kappa}T\} + X_S}{\exp\{\sigma\sqrt{2\kappa}T\} - \exp\{-\sigma\sqrt{2\kappa}T\}}\right)\sigma\sqrt{2\kappa} \tag{142}$$

$$B\sigma\sqrt{2\kappa} = \left(\frac{x_a\exp\{\sigma\sqrt{2\kappa}T\} - x_b - X_S}{\exp\{2\sigma\sqrt{2\kappa}T\} - 1}\right)\sigma\sqrt{2\kappa} \tag{143}$$

Then, we get

$$x_b\dot{x}(t_b)-x_a\dot{x}(t_a)=\sigma\sqrt{2\kappa}\left[\frac{(x_a^2+x_b^2)\cosh\sigma\sqrt{2\kappa}T-2x_ax_b}{\sinh\sigma\sqrt{2\kappa}T}-\frac{\left(x_a-x_b\cosh\sigma\sqrt{2\kappa}T\right)X_S}{\sinh\sigma\sqrt{2\kappa}T}\right]-x_bX_C \tag{144}$$

Next, compute the second component, we get

$$\begin{aligned}
I&=\sigma^2\int_{t_a}^{t_b}\left[f(t)-\frac{\mu'(t)}{\sigma^2}\right]x(t)dt\\
&=\frac{\sigma^2\left(x_b-x_a\exp\{-\sigma\sqrt{2\kappa}T\}+X_S\right)}{2\sinh\sigma\sqrt{2\kappa}T}\int_{t_a}^{t_b}\left[f(t)-\frac{\mu'(t)}{\sigma^2}\right]\exp\{\sigma\sqrt{2\kappa}(t-t_a)\}dt\\
&+\frac{\sigma^2\left(x_a-(x_b+X_S)\exp\{-\sigma\sqrt{2\kappa}T\}\right)}{2\sinh\sigma\sqrt{2\kappa}T}\int_{t_a}^{t_b}\left[f(t)-\frac{\mu'(t)}{\sigma^2}\right]\exp\{\sigma\sqrt{2\kappa}(t_b-t)\}dt\\
&-\frac{\sigma^3}{\sqrt{2\kappa}}\int_{t_a}^{t_b}\int_{t_a}^{t}\left[f(t)-\frac{\mu'(t)}{\sigma^2}\right]\left[f(s)-\frac{\mu'(s)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t-s)dsdt
\end{aligned} \tag{145}$$

Finally, we get

$$\begin{aligned}
x_b\dot{x}(t_b)-x_a\dot{x}(t_a)&=\sigma\sqrt{2\kappa}\left[\frac{(x_a^2+x_b^2)\cosh\sigma\sqrt{2\kappa}T-2x_ax_b}{\sinh\sigma\sqrt{2\kappa}T}\right]\\
&-\sigma\sqrt{2\kappa}\left[\frac{\left(x_a-x_b\cosh\sigma\sqrt{2\kappa}T\right)\left(\frac{\sigma}{\sqrt{2\kappa}}\int_{t_a}^{t_b}\left[f(s)-\frac{\mu'(s)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t_b-s)ds\right)}{\sinh\sigma\sqrt{2\kappa}T}\right]\\
&-\sigma^2x_b\int_{t_a}^{t_b}\left[f(s)-\frac{\mu'(s)}{\sigma^2}\right]\cos\sigma\sqrt{2\kappa}(t_b-s)ds
\end{aligned} \tag{146}$$

$$
\begin{aligned}
I' &= \sigma^2 \int_{t_a}^{t_b} \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] x(t) dt \\
&= \frac{\sigma^2 \Big(x_b - x_a \exp\{-\sigma\sqrt{2\kappa}T\} + \frac{\sigma}{\sqrt{2\kappa}} \int_{t_a}^{t_b} \Big[f(s) - \frac{\mu'(s)}{\sigma^2} \Big] \sin \sigma\sqrt{2\kappa}(t_b - s) ds \Big)}{2 \sinh \sigma\sqrt{2\kappa}T} \\
&\times \int_{t_a}^{t_b} \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \exp\{\sigma\sqrt{2\kappa}(t - t_a)\} dt \\
&+ \frac{\sigma^2 \Big(x_a - (x_b + \frac{\sigma}{\sqrt{2\kappa}} \int_{t_a}^{t_b} \Big[f(s) - \frac{\mu'(s)}{\sigma^2} \Big] \sin \sigma\sqrt{2\kappa}(t_b - s) ds) \exp\{-\sigma\sqrt{2\kappa}T\} \Big)}{2 \sinh \sigma\sqrt{2\kappa}T} \\
&\times \int_{t_a}^{t_b} \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \exp\{\sigma\sqrt{2\kappa}(t_b - t)\} dt \\
&- \frac{\sigma^3}{\sqrt{2\kappa}} \int_{t_a}^{t_b} \int_{t_a}^{t} \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \left[f(s) - \frac{\mu'(s)}{\sigma^2} \right] \sin \sigma\sqrt{2\kappa}(t - s) ds dt
\end{aligned}
\tag{147}
$$

Hence, S_{cl} can be written as

$$
\begin{aligned}
S_{cl} &= \frac{1}{2}\sigma\sqrt{2\kappa} \left[\frac{(x_a^2 + x_b^2) \cosh \sigma\sqrt{2\kappa}T - 2x_a x_b}{\sinh \sigma\sqrt{2\kappa}T} \right] \\
&- \frac{1}{2}\sigma\sqrt{2\kappa} \left[\frac{\Big(x_a - x_b \cosh \sigma\sqrt{2\kappa}T \Big) \Big(\frac{\sigma}{\sqrt{2\kappa}} \int_{t_a}^{t_b} \Big[f(s) - \frac{\mu'(s)}{\sigma^2} \Big] \sin \sigma\sqrt{2\kappa}(t_b - s) ds \Big)}{\sinh \sigma\sqrt{2\kappa}T} \right] \\
&- \frac{1}{2}\sigma^2 x_b \int_{t_a}^{t_b} \left[f(s) - \frac{\mu'(s)}{\sigma^2} \right] \cos \sigma\sqrt{2\kappa}(t_b - s) ds \\
&- \frac{\sigma^2 \Big(x_b - x_a \exp\{-\sigma\sqrt{2\kappa}T\} + \frac{\sigma}{\sqrt{2\kappa}} \int_{t_a}^{t_b} \Big[f(s) - \frac{\mu'(s)}{\sigma^2} \Big] \sin \sigma\sqrt{2\kappa}(t_b - s) ds \Big)}{4 \sinh \sigma\sqrt{2\kappa}T} \\
&\times \int_{t_a}^{t_b} \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \exp\{\sigma\sqrt{2\kappa}(t - t_a)\} dt \\
&- \frac{\sigma^2 \Big(x_a - (x_b + \frac{\sigma}{\sqrt{2\kappa}} \int_{t_a}^{t_b} \Big[f(s) - \frac{\mu'(s)}{\sigma^2} \Big] \sin \sigma\sqrt{2\kappa}(t_b - s) ds) \exp\{-\sigma\sqrt{2\kappa}T\} \Big)}{4 \sinh \sigma\sqrt{2\kappa}T} \\
&\times \int_{t_a}^{t_b} \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \exp\{\sigma\sqrt{2\kappa}(t_b - t)\} dt \\
&+ \frac{1}{2} \frac{\sigma^3}{\sqrt{2\kappa}} \int_{t_a}^{t_b} \int_{t_a}^{t} \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \left[f(s) - \frac{\mu'(s)}{\sigma^2} \right] \sin \sigma\sqrt{2\kappa}(t - s) ds dt
\end{aligned}
\tag{148}
$$

The kernel is thus calculated as

$$
\begin{aligned}
K(b,a) = {} & \left(\frac{\sqrt{2\kappa}\sigma}{2\pi\sigma^2\sinh\sqrt{2\kappa}\sigma(t_b-t_a)}\right)^{1/2}\exp\left(-\frac{1}{2\sigma^2}\int_{t_a}^{t_b}\mu^2(t)dt\right) \\
& \times\exp\left(-\frac{1}{4\kappa}\int_{t_a}^{t_b}f^2(t)dt\right)\exp\{-\frac{1}{\sigma^2}S_{cl}\}
\end{aligned}
\tag{149}
$$

□

Hence,

$$
\begin{aligned}
K(x,t;y,0) = {} & \left(\frac{\sqrt{2\kappa}\sigma}{2\pi\sigma^2\sinh\sqrt{2\kappa}\sigma t}\right)^{1/2}\exp\left(-\frac{1}{2\sigma^2}\int_0^t\mu^2(\tau)d\tau\right) \\
& \times\exp\left(-\frac{1}{4\kappa}\int_0^t f^2(\tau)d\tau\right)\exp\{-\frac{1}{\sigma^2}S_{cl}(x,y,t)\}
\end{aligned}
\tag{150}
$$

where $S_{cl}(x,y,t)$ is given by

$$
\begin{aligned}
S_{cl}(x,y,t) = {} & \frac{1}{2}\sigma\sqrt{2\kappa}\left[\frac{(x^2+y^2)\cosh\sigma\sqrt{2\kappa}t-2xy}{\sinh\sigma\sqrt{2\kappa}t}\right] \\
& -\frac{1}{2}\sigma\sqrt{2\kappa}\left[\frac{\left(y-x\cosh\sigma\sqrt{2\kappa}t\right)\left(\frac{\sigma}{\sqrt{2\kappa}}\int_0^t\left[f(s)-\frac{\mu'(s)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t-s)ds\right)}{\sinh\sigma\sqrt{2\kappa}t}\right] \\
& -\frac{1}{2}\sigma^2x\int_0^t\left[f(s)-\frac{\mu'(s)}{\sigma^2}\right]\cos\sigma\sqrt{2\kappa}(t-s)ds \\
& -\frac{\sigma^2\left(x-y\exp\{-\sigma\sqrt{2\kappa}t\}+\frac{\sigma}{\sqrt{2\kappa}}\int_0^t\left[f(s)-\frac{\mu'(s)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t-s)ds\right)}{4\sinh\sigma\sqrt{2\kappa}t} \\
& \times\int_0^t\left[f(s)-\frac{\mu'(s)}{\sigma^2}\right]\exp\{\sigma\sqrt{2\kappa}s\}ds \\
& -\frac{\sigma^2\left(y-(x+\frac{\sigma}{\sqrt{2\kappa}}\int_0^t\left[f(s)-\frac{\mu'(s)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(t-s)ds)\exp\{-\sigma\sqrt{2\kappa}t\}\right)}{4\sinh\sigma\sqrt{2\kappa}t} \\
& \times\int_0^t\left[f(s)-\frac{\mu'(s)}{\sigma^2}\right]\exp\{\sigma\sqrt{2\kappa}(t-s)\}ds \\
& +\frac{1}{2}\frac{\sigma^3}{\sqrt{2\kappa}}\int_0^t\int_0^\tau\left[f(\tau)-\frac{\mu'(\tau)}{\sigma^2}\right]\left[f(s)-\frac{\mu'(s)}{\sigma^2}\right]\sin\sigma\sqrt{2\kappa}(\tau-s)dsd\tau
\end{aligned}
$$

A.5 Proof of Proposition 2

Proof. Taking first order condition, we get

$$\frac{\partial c(l(x,t))}{\partial l} = v(x,t) - v(x^*(t),t) \tag{151}$$

Let

$$U(x,t) = v(x,t) - v(x^*(t),t) \tag{152}$$

and

$$u(x,t) = \frac{\partial U}{\partial x} \tag{153}$$

then we have

$$u(x,t) = \frac{\partial U}{\partial x} = \frac{\partial v}{\partial x} \tag{154}$$

Hence, we get

$$\frac{\partial c(l(x,t))}{\partial l} = U(x,t) \tag{155}$$

Note that

$$\frac{\partial c(l(x,t))}{\partial x} = \frac{\partial c(l(x,t))}{\partial l}\frac{\partial l}{\partial x} = U(x,t)\frac{\partial l}{\partial x} \tag{156}$$

Take l its optimal value $l^*(x,t)$ and take the derivative twice, we get

$$\frac{\partial^2 c(l^*(x,t))}{\partial x^2} = u(x,t)\frac{\partial l^*}{\partial x} + U(x,t)\frac{\partial^2 l^*}{\partial x^2} \tag{157}$$

Since the optimal $l^*(x,t)$ is the generalized hazard function $\Lambda(x,t)$, it follows that

$$l^*(x,t) = \Lambda(x,t) \tag{158}$$

hence we have

$$\frac{\partial c(\Lambda(x,t))}{\partial x} = U(x,t)\frac{\partial \Lambda(x,t)}{\partial x} \tag{159}$$

and

$$\frac{\partial^2 c(\Lambda(x,t))}{\partial x^2} = u(x,t)\frac{\partial \Lambda(x,t)}{\partial x} + U(x,t)\frac{\partial^2 \Lambda(x,t)}{\partial x^2} \tag{160}$$

It follows from the definition of U and the optimal reinjection point x^* satisfying the boundary condition at $x^*(t)$ as $u(x^*,t) = U(x^*,t) = 0$ that a necessary condition for $\Lambda(x,t)$ to be rationalized as a generalized hazard function is

$$\frac{\partial c(\Lambda(x^*,t))}{\partial x} = \frac{\partial^2 c(\Lambda(x^*,t))}{\partial x^2} = 0 \tag{161}$$

which determines the optimal reinjection point $x^*(t)$. Later we will see that $x^*(t)$ can be explicitly determined in terms of the parameters included in the generalized hazard function $\Lambda(x,t)$, given an explicit functional form of $\Lambda(x,t)$.

Moreover, after plugging the optimal policy $l^* = \Lambda(x,t)$ into the HJB equation, we obtain

$$\begin{aligned} r(t)v(x,t) = Bx^2 + \mu(t)\frac{\partial v(x,t)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 v(x,t)}{\partial x^2} \\ \Lambda(x,t)(v(x^*(t),t) - v(x,t)) + c(\Lambda(x,t)) + \frac{\partial v(x,t)}{\partial t} \end{aligned} \tag{162}$$

Take envelop condition to above equation (treat $\Lambda(x,t)$ as given for the envelop theorem since it is the optimal solution of the HJB), we get

$$(r(t)+\Lambda)u(x,t) = 2Bx + \mu(t)\frac{\partial u}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} \tag{163}$$

and the value function $v(x,t)$ can be generally expressed as

$$v(x,t) = \frac{B}{r(t)}(x^*)^2 + \frac{\sigma^2}{2r(t)}\frac{\partial u(x^*,t)}{\partial x} + \frac{1}{r(t)}\frac{\partial v(x^*,t)}{\partial t} + \int_{x^*}^{x} u(z,t)dz \tag{164}$$

for $x \in (x^*, \infty)$, and

$$v(x,t) = \frac{B}{r(t)}(x^*)^2 + \frac{\sigma^2}{2r(t)}\frac{\partial u(x^*,t)}{\partial x} + \frac{1}{r(t)}\frac{\partial v(x^*,t)}{\partial t} + \int_{x}^{x^*} u(z,t)dz \tag{165}$$

for $x \in (-\infty, x^*)$.

□

A.6 Equivalence of KFE and Path Integrals with Time-Varying Inflation

Proof. Let $\Lambda^{\psi}(x,t) = \kappa x^2 - \left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x + \frac{f^2(t)}{4\kappa} + \frac{1}{2}\left[\frac{\mu(t)}{\sigma}\right]^2$, then we have

$$\psi(x,t) = \int_{-\infty}^{\infty} K(x,t;y,0)\psi(y,0)dy \tag{166}$$

Note that for a short time interval ϵ, above equation can be rewritten as

$$\psi(x,t+\epsilon) = \frac{1}{A}\int_{-\infty}^{\infty} \exp\{-\frac{1}{\sigma^2}\epsilon L\left(\frac{x-y}{\epsilon}, \frac{x+y}{2}\right)\}\psi(y,t)dy \tag{167}$$

where $L = \frac{1}{2}\dot{x}^2 + \sigma^2\Lambda^\psi(x,t)$. And it can be rewritten as

$$\begin{aligned}\psi(x,t+\epsilon) = \frac{1}{A}\int_{-\infty}^{\infty} &\exp\{-\frac{1}{\sigma^2}\frac{(x-y)^2}{2\epsilon}\} \\ &\times \exp\{-\frac{1}{\sigma^2}\epsilon\sigma^2\Lambda^\psi\left(\frac{x+y}{2}, t\right)\}\psi(y,t)dy\end{aligned} \tag{168}$$

By a change of variables $y = x + \eta$, we have

$$\begin{aligned}\psi(x,t+\epsilon) = \frac{1}{A}\int_{-\infty}^{\infty} &\exp\{-\frac{\eta^2}{2\sigma^2\epsilon}\} \\ &\times \exp\{-\frac{1}{\sigma^2}\epsilon\sigma^2\Lambda^\psi\left(x+\frac{\eta}{2}, t\right)\}\psi(x+\eta,t)d\eta\end{aligned} \tag{169}$$

By expanding ψ in a power series, we get

$$\begin{aligned}\psi(x,t) + \epsilon\frac{\partial\psi}{\partial t} = \frac{1}{A}\int_{-\infty}^{\infty} &\exp\{-\frac{\eta^2}{2\sigma^2\epsilon}\} \\ \times &\left[1 - \frac{1}{\sigma^2}\epsilon\sigma^2\Lambda^\psi(x,t)\right]\left[\psi(x,t) + \eta\frac{\partial\psi}{\partial x} + \frac{\eta^2}{2}\frac{\partial^2\psi}{\partial x^2}\right]d\eta\end{aligned} \tag{170}$$

Note that the leading term on the right-hand side is equal to (by Gaussian integral)

$$\frac{1}{A}\int_{-\infty}^{\infty} \exp\{-\frac{\eta^2}{2\sigma^2\epsilon}\}d\eta = \frac{1}{A}\left(2\pi\sigma^2\epsilon\right)^{1/2} \tag{171}$$

On the left-hand side, there is only $\psi(x,t)$, therefore, to let both sides agree to each other, A must be chosen so that $\frac{1}{A}(2\pi\sigma^2\epsilon)^{1/2} = 1$, that is,

$$A = \left(2\pi\sigma^2\epsilon\right)^{1/2} \tag{172}$$

Moreover, we can calculate the other two terms on the right-hand side of the expanded equation, that is,

$$\frac{1}{A}\int_{-\infty}^{\infty} \eta\exp\{-\frac{\eta^2}{2\sigma^2\epsilon}\}d\eta = 0 \tag{173}$$

$$\frac{1}{A}\int_{-\infty}^{\infty}\eta^2\exp\{-\frac{\eta^2}{2\sigma^2\epsilon}\}d\eta = \sigma^2\epsilon \tag{174}$$

Finally, writing out the full version of the expanded equation using the fact that the second order of ϵ goes to zero, that is, $\epsilon^2 \to 0$, we get

$$\psi(x,t) + \epsilon\frac{\partial\psi}{\partial t} = \psi(x,t) - \frac{1}{\sigma^2}\epsilon\sigma^2\Lambda^{\psi}(x,t)\psi(x,t) + \frac{\sigma^2\epsilon}{2}\frac{\partial^2\psi}{\partial x^2} \tag{175}$$

Simplifying it, we get

$$\frac{\partial\psi}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2\psi}{\partial x^2} - \Lambda^{\psi}(x,t)\psi(x,t) \tag{176}$$

where $\Lambda^{\psi}(x,t) = \kappa x^2 - \left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x + \frac{f^2(t)}{4\kappa} + \frac{1}{2}\left[\frac{\mu(t)}{\sigma}\right]^2$, which is the KFE. Hence, we have proven the equivalence of path integral formulation and the KFE formulation. □

A.7 Proof of Proposition 5

Define $L(\dot{x},x,t) = \frac{1}{2}\dot{x}^2 - \sigma^2\Lambda(x,t)$, where σ^2 is the variance of the price gap x as a standard Brownian motion for the uncontrolled price process. Then,

$$L(\dot{x},x,t) = \frac{1}{2}\dot{x}^2 - \kappa\sigma^2x^2 + \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x - \frac{\sigma^2 f^2(t)}{4\kappa} - \frac{1}{2}\mu^2(t)$$

and we have, by defining $S[x(t)] = \int_{t_a}^{t_b} L(\dot{x},x,t)dt$,

$$S[x(t)] = S[\bar{x}(t) + y(t)]$$

that is,

$$
\begin{aligned}
I &= S[x(t)] \\
&= \int_{t_a}^{t_b} \left(\frac{1}{2}\left(\dot{\bar{x}}^2 + 2\dot{\bar{x}}\dot{y} + \dot{y}^2\right) - \kappa\sigma^2(\bar{x}(t) + y(t))^2 + \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] (\bar{x}(t) + y(t)) - \frac{\sigma^2 f^2(t)}{4\kappa} - \frac{\mu^2(t)}{2} \right) dt \\
&= \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) - \kappa\sigma^2\bar{x}^2(t) + \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \bar{x}(t) \right) dt \\
&+ \int_{t_a}^{t_b} \left(\dot{\bar{x}}(t)\dot{y}(t) + \frac{1}{2}\dot{y}^2(t) - 2\kappa\sigma^2\bar{x}(t)y(t) - \kappa\sigma^2 y^2(t) + \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] y(t) \right) dt \\
&- \frac{\sigma^2}{4\kappa} \int_{t_a}^{t_b} f^2(t)dt - \frac{1}{2} \int_{t_a}^{t_b} \mu^2(t)dt
\end{aligned}
$$

Note that

$$
\begin{aligned}
S_1 &= \int_{t_a}^{t_b} \left(\dot{\bar{x}}(t)\dot{y}(t) - 2\kappa\sigma^2\bar{x}(t)y(t) + \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] y(t) \right) dt \\
&= \int_{t_a}^{t_b} \dot{\bar{x}}(t)dy(t) - 2\kappa\sigma^2 \int_{t_a}^{t_b} \bar{x}(t)y(t)dt + \int_{t_a}^{t_b} \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] y(t)dt \\
&= [\dot{\bar{x}}(t)y(t)]_{t_a}^{t_b} - \int_{t_a}^{t_b} \ddot{\bar{x}}(t)y(t)dt - 2\kappa\sigma^2 \int_{t_a}^{t_b} \bar{x}(t)y(t)dt + \int_{t_a}^{t_b} \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] y(t)dt \\
&= - \int_{t_a}^{t_b} \left(-2\kappa\sigma^2\bar{x}(t) + \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \right) y(t)dt - 2\kappa\sigma^2 \int_{t_a}^{t_b} \bar{x}(t)y(t)dt + \int_{t_a}^{t_b} \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] y(t)dt \\
&= 0
\end{aligned}
$$

where we have used $y(t_a) = y(t_b) = 0$ and from Euler Lagrange equation for $L(\dot{x}, x, t) = \frac{1}{2}\dot{x}^2 - \kappa\sigma^2 x^2 + \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] x - \frac{\sigma^2 f^2(t)}{4\kappa} - \frac{1}{2}\mu^2(t)$ to get $\ddot{\bar{x}}(t) = -2\kappa\sigma^2\bar{x}(t) + \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right]$.

Therefore, we get

$$
\begin{aligned}
S[x(t)] &= S[\bar{x}(t) + y(t)] \\
&= \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) - \kappa\sigma^2\bar{x}^2(t) + \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \bar{x}(t) \right) dt + \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{y}^2(t) - \kappa\sigma^2 y^2(t) \right) dt \\
&- \frac{\sigma^2}{4\kappa} \int_{t_a}^{t_b} f^2(t)dt - \frac{1}{2} \int_{t_a}^{t_b} \mu^2(t)dt \\
&= S[\bar{x}(t)] + S[y(t)] - \frac{\sigma^2}{4\kappa} \int_{t_a}^{t_b} f^2(t)dt - \frac{1}{2} \int_{t_a}^{t_b} \mu^2(t)dt
\end{aligned}
$$

where

$$
S[\bar{x}(t)] = \int_{t_a}^{t_b} \left(\frac{1}{2}\dot{\bar{x}}^2(t) - \kappa\sigma^2\bar{x}^2(t) + \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \bar{x}(t) \right) dt
$$

$$S[y(t)] = \int_{t_a}^{t_b} \left(\frac{1}{2} \dot{y}^2(t) - \kappa \sigma^2 y^2(t) \right) dt$$

Therefore, we finally get

$$\begin{aligned}
K(b,a) &= \int_a^b \exp\left(\frac{i}{\sigma^2} S[x(t)] \right) \mathcal{D}x(t) \\
&= \int_0^0 \exp\left(\frac{i}{\sigma^2} S[\bar{x}(t) + y(t)] - \frac{i}{4\kappa} \int_{t_a}^{t_b} f^2(t)dt - \frac{i}{2\sigma^2} \int_{t_a}^{t_b} \mu^2(t)dt \right) \mathcal{D}y(t) \\
&= \int_0^0 \exp\left(\frac{i}{\sigma^2} S[\bar{x}(t)] + \frac{i}{\sigma^2} S[y(t)] - \frac{i}{4\kappa} \int_{t_a}^{t_b} f^2(t)dt - \frac{i}{2\sigma^2} \int_{t_a}^{t_b} \mu^2(t)dt \right) \mathcal{D}y(t) \\
&= \exp\left(-\frac{i}{4\kappa} \int_{t_a}^{t_b} f^2(t)dt \right) \exp\left(-\frac{i}{2\sigma^2} \int_{t_a}^{t_b} \mu^2(t)dt \right) \\
&\times \exp\left(\frac{i}{\sigma^2} S[\bar{x}(t)] \right) \int_0^0 \exp\left(\frac{i}{\sigma^2} S[y(t)] \right) \mathcal{D}y(t)
\end{aligned}$$

That is, given the generalized hazard function with transitional inflation, the corresponding kernel is given by

$$\begin{aligned}
K(b,a) = \exp&\left(-\frac{i}{2\sigma^2} \int_{t_a}^{t_b} \mu^2(t)dt \right) \exp\left(-\frac{i}{4\kappa} \int_{t_a}^{t_b} f^2(t)dt \right) \\
&\times \exp\left(\frac{i}{\sigma^2} S[\bar{x}(t)] \right) \int_0^0 \exp\left(\frac{i}{\sigma^2} S[y(t)] \right) \mathcal{D}y(t)
\end{aligned}$$

First, we can compute $\int_0^0 \exp\left(\frac{i}{\sigma^2} S[y(t)] \right) \mathcal{D}y(t)$ using the Fourier series method, and it turns out (the same as in the case where $\mu(t) = f(t) = 0$)

$$\begin{aligned}
\int_0^0 \exp\left(\frac{i}{\sigma^2} S[y(t)] \right) \mathcal{D}y(t) &= \int_0^0 \exp\left(\frac{i}{\sigma^2} \int_{t_a}^{t_b} \left(\frac{1}{2} \dot{y}^2(t) + \kappa \sigma^2 y^2(t) \right) dt \right) \mathcal{D}y(t) \\
&= \left(\frac{\sqrt{2\kappa}\sigma}{2\pi i \sigma^2 \sin \sqrt{2\kappa}\sigma(t_b - t_a)} \right)^{1/2}
\end{aligned}$$

Next, we compute

$$\exp\left(\frac{i}{\sigma^2} \int_{t_a}^{t_b} \left(\frac{1}{2} \dot{\bar{x}}^2(t) - \kappa \sigma^2 \bar{x}^2(t) + \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \bar{x}(t) \right) dt \right)$$

Note that since $\bar{x}$ can be any x due to that fact that $\bar{x}$ is just any arbitrary subset of x depending on our choice, it follows that $\bar{x}$ and x can be used interchangeably, i.e., $\bar{x} = x$. Therefore, computing

$$\exp\left(\frac{i}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{\bar{x}}^2(t) - \kappa\sigma^2\bar{x}^2(t) + \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]\bar{x}(t)\right)dt\right)$$

is equivalent to computing

$$\exp\left(\frac{i}{\sigma^2}\int_{t_a}^{t_b}\left(\frac{1}{2}\dot{x}^2(t) - \kappa\sigma^2 x^2(t) + \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x(t)\right)dt\right)$$

From Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

associated with the $L = \frac{1}{2}\dot{x}^2(t) - \kappa\sigma^2 x^2(t) + \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x(t) - \frac{\sigma^2 f^2(t)}{4\kappa} - \frac{1}{2}\mu^2(t)$ we get

$$\frac{d\dot{x}}{dt} + 2\kappa\sigma^2 x - \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right] = 0,$$

or equivalently,

$$\ddot{x} = -2\kappa\sigma^2 x + \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]$$

which is an inhomogeneous linear second-order ODE whose solution can be written as

$$x(t) = A\sin\sqrt{2\kappa}\sigma(t-t_a) + B\sin\sqrt{2\kappa}\sigma(t_b - t) + \frac{1}{\sqrt{2\kappa}\sigma}\int_{t_a}^{t}\sigma^2\left[f(s) - \frac{\mu'(s)}{\sigma^2}\right]\sin\sqrt{2\kappa}\sigma(t-s)ds,$$

and

$$\dot{x}(t)$$

$$= \sqrt{2\kappa}\sigma\left(A\cos\sqrt{2\kappa}\sigma(t-t_a) - B\cos\sqrt{2\kappa}\sigma(t_b - t) + \frac{1}{\sqrt{2\kappa}\sigma}\int_{t_a}^{t}\sigma^2\left[f(s) - \frac{\mu'(s)}{\sigma^2}\right]\cos\sqrt{2\kappa}\sigma(t-s)ds\right)$$

Therefore,

$$S_{cl} = \int_{t_a}^{t_b}\left(\frac{1}{2}\dot{x}^2(t) - \frac{1}{2}(2\kappa\sigma^2)x^2(t) + \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x(t)\right)dt$$

$$= \frac{1}{2}\int_{t_a}^{t_b}\dot{x}^2(t)dt - \frac{1}{2}\int_{t_a}^{t_b}(2\kappa\sigma^2)x^2(t)dt + \int_{t_a}^{t_b}\sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x(t)dt$$

$$= \frac{1}{2}\left([x\dot{x}]_{t_a}^{t_b} - \int_{t_a}^{t_b} x\ddot{x}dt\right) - \frac{1}{2}\int_{t_a}^{t_b}(2\kappa\sigma^2)x^2(t)dt + \int_{t_a}^{t_b}\sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x(t)dt$$

$$= \frac{1}{2}\left([x\dot{x}]_{t_a}^{t_b} - \int_{t_a}^{t_b} x\left(-(2\kappa\sigma^2)x + \sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]\right)dt\right)$$

$$-\frac{1}{2}\int_{t_a}^{t_b}(2\kappa\sigma^2)x^2(t)dt + \int_{t_a}^{t_b}\sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x(t)dt$$

$$= \frac{1}{2}[x(t)\dot{x}(t)]_{t_a}^{t_b} + \frac{1}{2}\int_{t_a}^{t_b}\sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]x(t)dt$$

$$= \frac{1}{2}\times$$

$$[x(t)\sqrt{2\kappa}\sigma[A\cos\sqrt{2\kappa}\sigma(t-t_a) - B\cos\sqrt{2\kappa}\sigma(t_b-t) + \frac{1}{\sqrt{2\kappa}\sigma}\int_{t_a}^{t}\sigma^2\left[f(s) - \frac{\mu'(s)}{\sigma^2}\right]\cos\sqrt{2\kappa}\sigma(t-s)ds]]_{t_a}^{t_b}$$

$$+\frac{1}{2}\times$$

$$\int_{t_a}^{t_b}\sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]\times$$

$$\left(A\sin\sqrt{2\kappa}\sigma(t-t_a) + B\sin\sqrt{2\kappa}\sigma(t_b-t) + \frac{1}{\sqrt{2\kappa}\sigma}\int_{t_a}^{t}\sigma^2\left[f(s) - \frac{\mu'(s)}{\sigma^2}\right]\sin\sqrt{2\kappa}\sigma(t-s)ds\right)dt$$

$$= \frac{\sqrt{2\kappa}\sigma}{2\sin\sqrt{2\kappa}\sigma T}\times$$

$$\left(\left(x_b^2 + x_a^2\right)\cos\sqrt{2\kappa}\sigma T - 2x_b x_a\right)$$

$$+\frac{\sqrt{2\kappa}\sigma}{2\sin\sqrt{2\kappa}\sigma T}\times$$

$$\left(\frac{2x_b}{\sqrt{2\kappa}\sigma}\int_{t_a}^{t_b}\sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]\sin\sqrt{2\kappa}\sigma(t-t_a)dt + \frac{2x_a}{\sqrt{2\kappa}\sigma}\int_{t_a}^{t_b}\sigma^2\left[f(t) - \frac{\mu'(t)}{\sigma^2}\right]\sin\sqrt{2\kappa}\sigma(t_b-t)dt\right)$$

$$-\left(\frac{\sqrt{2\kappa}\sigma}{2\sin\sqrt{2\kappa}\sigma T}\right)\left(\frac{2}{2\kappa\sigma^2}\right)$$

$$\times \int_{t_a}^{t_b} \int_{t_a}^{t} \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \sigma^2 \left[f(s) - \frac{\mu'(s)}{\sigma^2} \right] \sin \sqrt{2\kappa}\sigma(t_b - t) \sin \sqrt{2\kappa}\sigma(s - t_a) ds dt$$

The kernel is thus given by

$$K(x_b, t_b; x_a, t_a) = \left(\frac{\sqrt{2\kappa}\sigma}{2\pi i \sigma^2 \sin \sqrt{2\kappa}\sigma(t_b - t_a)} \right)^{1/2} \exp\left(-\frac{i}{2\sigma^2} \int_{t_a}^{t_b} \mu^2(t) dt \right) \exp\left(-\frac{i}{4\kappa} \int_{t_a}^{t_b} f^2(t) dt \right)$$

$$\times \exp\left(\frac{i\sqrt{2\kappa}\sigma}{2\sigma^2 \sin \sqrt{2\kappa}\sigma(t_b - t_a)} \left((x_b^2 + x_a^2) \cos \sqrt{2\kappa}\sigma(t_b - t_a) - 2x_b x_a \right) \right)$$

$$\times \exp\left(\frac{i\sqrt{2\kappa}\sigma}{2\sigma^2 \sin \sqrt{2\kappa}\sigma(t_b - t_a)} \frac{2x_b}{\sqrt{2\kappa}\sigma} \int_{t_a}^{t_b} \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \sin \sqrt{2\kappa}\sigma(t - t_a) dt \right)$$

$$\times \exp\left(\frac{i\sqrt{2\kappa}\sigma}{2\sigma^2 \sin \sqrt{2\kappa}\sigma(t_b - t_a)} \frac{2x_a}{\sqrt{2\kappa}\sigma} \int_{t_a}^{t_b} \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \sin \sqrt{2\kappa}\sigma(t_b - t) dt \right)$$

$$\times \exp\left(\frac{-i\sqrt{2\kappa}\sigma \int_{t_a}^{t_b} \int_{t_a}^{t} \sigma^4 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \left[f(s) - \frac{\mu'(s)}{\sigma^2} \right] \sin \sqrt{2\kappa}\sigma(t_b - t) \sin \sqrt{2\kappa}\sigma(s - t_a) ds dt}{2\sigma^2 \sin \sqrt{2\kappa}\sigma(t_b - t_a)} \frac{1}{\kappa\sigma^2} \right)$$

Next, we are ready to solve for G_{mn} by writing

$$\int \int \chi^*(x_b, t_b) K(x_b, t_b; x_a, t_a) \psi(x_a, t_a) dx_a dx_b = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} G_{mn} \phi_m^*(b) \phi_n(a) e^{-(i/\sigma^2)\lambda_m(t_b - t_a)},$$

where $K(x_b, t_b; x_a, t_a)$ is given above and $\psi(x_a, t_a)$ and $\chi^*(x_b, t_b)$ are given by $\psi(x) = \left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2} \right)^{1/4} e^{-(\sqrt{\kappa}\sigma/2\sigma^2)(x-a)^2}$ and $\chi(x) = \left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2} \right)^{1/4} e^{-(\sqrt{\kappa}\sigma/2\sigma^2)(x-b)^2}$, respectively, and λ_m and $\phi_n(x)$ are given by

$$-\lambda_m = -\sqrt{\kappa}\sigma^3 \left(m + \frac{1}{2} \right)$$

$$\phi_n(x) = \frac{1}{(2^n n!)^{1/2}} \left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2} \right)^{1/4} H_n\left(\sqrt{\frac{\sqrt{\kappa}\sigma}{\sigma^2}} x \right) e^{-\left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2} \right) x^2}$$

where H is the (physicist's) Hermite polynomial of degree n given by $H_n(x) =$

$(-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$ and $n = 0, 1, 2, ...$, that is, $\phi_n(a)$ and $\phi_m(b)$ are given by

$$\phi_n(a) = \frac{1}{(2^n n!)^{1/2}} \left(\frac{\sqrt{\kappa\sigma}}{\pi\sigma^2} \right)^{1/4} H_n\left(\sqrt{\frac{\sqrt{\kappa\sigma}}{\sigma^2}} a \right) e^{-\left(\frac{\sqrt{\kappa\sigma}}{2\sigma^2}\right) a^2}$$

$$\phi_m(b) = \frac{1}{(2^m m!)^{1/2}} \left(\frac{\sqrt{\kappa\sigma}}{\pi\sigma^2} \right)^{1/4} H_m\left(\sqrt{\frac{\sqrt{\kappa\sigma}}{\sigma^2}} b \right) e^{-\left(\frac{\sqrt{\kappa\sigma}}{2\sigma^2}\right) b^2}$$

Plugging in all those expressions, we get

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{\sqrt{\kappa\sigma}}{\pi\sigma^2} \right)^{1/4} e^{-(\sqrt{\kappa\sigma}/2\sigma^2)(x_b - b)^2} K(x_b, t_b; x_a, t_a) \left(\frac{\sqrt{\kappa\sigma}}{\pi\sigma^2} \right)^{1/4} e^{-(\sqrt{\kappa\sigma}/2\sigma^2)(x_a - a)^2} dx_a dx_b$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} G_{mn} \left(\frac{\sqrt{\kappa\sigma}}{2\sigma^2} \right)^{\frac{m}{2}} \frac{b^m}{\sqrt{m!}} e^{-\frac{\sqrt{\kappa\sigma}}{4\sigma^2} b^2} \left(\frac{\sqrt{\kappa\sigma}}{2\sigma^2} \right)^{\frac{n}{2}} \frac{a^n}{\sqrt{n!}} e^{-\frac{\sqrt{\kappa\sigma}}{4\sigma^2} a^2} e^{-(i/\sigma^2)\left(\sqrt{\kappa\sigma}\sigma^2\left(m + \frac{1}{2}\right)\right)(t_b - t_a)},$$

Perform the double Gaussian integral on the left hand side, which is a bit lengthy but direct, and simplify the term on the right hand side, we get the equation above rewritten as

$$\begin{aligned}
&\exp\left(-\frac{i}{2\sigma^2} \int_{t_a}^{t_b} \mu^2(t) dt \right) \exp\left(-\frac{i}{4\kappa} \int_{t_a}^{t_b} f^2(t) dt \right) \\
&\times \exp\left(-\frac{i\sqrt{\kappa\sigma}(t_b - t_a)}{2} - \frac{\sqrt{\kappa\sigma}}{4\sigma^2} \left(a^2 + b^2 - 2ab e^{-i\sqrt{\kappa\sigma}(t_b - t_a)} \right) \right) \\
&\times \exp\left(i\sqrt{\frac{\sqrt{\kappa\sigma}}{2\sigma^2}} \left(\frac{a}{\sqrt{2\sigma^2\sqrt{\kappa\sigma}}} \int_{t_a}^{t_b} \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] e^{-i\sqrt{\kappa\sigma}t} dt \right) \right) \\
&\times \exp\left(\left[\frac{b}{\sqrt{2\sigma^2\sqrt{\kappa\sigma}}} \int_{t_a}^{t_b} \sigma^2 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] e^{i\sqrt{\kappa\sigma}t} dt \right] e^{-i\sqrt{\kappa\sigma}(t_b - t_a)} \right) \\
&\div \exp\left(\frac{1}{2\sigma^2\sqrt{\kappa\sigma}} \int_{t_a}^{t_b} \int_{t_a}^{t} \sigma^4 \left[f(t) - \frac{\mu'(t)}{\sigma^2} \right] \left[f(s) - \frac{\mu'(s)}{\sigma^2} \right] e^{-i\sqrt{\kappa\sigma}(t - s)} ds dt \right) \\
&= \exp\left(-\frac{\sqrt{\kappa\sigma}}{4\sigma^2} (b^2 + a^2) \right) \times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} G_{mn} \frac{b^m a^n}{\sqrt{m!n!}} \left(\frac{\sqrt{\kappa\sigma}}{2\sigma^2} \right)^{(m+n)/2} e^{-i(m+1/2)\sqrt{\kappa\sigma}(t_b - t_a)},
\end{aligned}$$

Note that the value of G_{00} can be obtained by setting $m = n = 0$ from

$$G_{mn} = e^{(i/\sigma^2)\lambda_m(t_b-t_a)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_m(x_b)K(x_b,t_b;x_a,t_a)\phi_n(x_a)dx_a dx_b$$

where

$$K(x_b,t_b;x_a,t_a) = \left(\frac{\sqrt{2\kappa}\sigma}{2\pi i\sigma^2 \sin\sqrt{2\kappa}\sigma(t_b-t_a)}\right)^{1/2} \exp\left(-\frac{i}{2\sigma^2}\int_{t_a}^{t_b}\mu^2(t)dt\right)\exp\left(-\frac{i}{4\kappa}\int_{t_a}^{t_b} f^2(t)dt\right)$$
$$\times\exp\left(\frac{i\sqrt{2\kappa}\sigma}{2\sigma^2\sin\sqrt{2\kappa}\sigma(t_b-t_a)}\left((x_b^2+x_a^2)\cos\sqrt{2\kappa}\sigma(t_b-t_a)-2x_bx_a\right)\right)$$
$$\times\exp\left(\frac{i\sqrt{2\kappa}\sigma}{2\sigma^2\sin\sqrt{2\kappa}\sigma(t_b-t_a)}\frac{2x_b}{\sqrt{2\kappa}\sigma}\int_{t_a}^{t_b}\sigma^2\left[f(t)-\frac{\mu'(t)}{\sigma^2}\right]\sin\sqrt{2\kappa}\sigma(t-t_a)dt\right)$$
$$\times\exp\left(\frac{i\sqrt{2\kappa}\sigma}{2\sigma^2\sin\sqrt{2\kappa}\sigma(t_b-t_a)}\frac{2x_a}{\sqrt{2\kappa}\sigma}\int_{t_a}^{t_b}\sigma^2\left[f(t)-\frac{\mu'(t)}{\sigma^2}\right]\sin\sqrt{2\kappa}\sigma(t_b-t)dt\right)$$
$$\times\exp\left(\frac{-i\sqrt{2\kappa}\sigma\int_{t_a}^{t_b}\int_{t_a}^{t}\sigma^4\left[f(t)-\frac{\mu'(t)}{\sigma^2}\right]\left[f(s)-\frac{\mu'(s)}{\sigma^2}\right]\sin\sqrt{2\kappa}\sigma(t_b-t)\sin\sqrt{2\kappa}\sigma(s-t_a)dsdt}{2\sigma^2\sin\sqrt{2\kappa}\sigma(t_b-t_a)}\frac{1}{\kappa\sigma^2}\right)$$

which is a gaussian integral and can be computed as

$$G_{00} = \exp\left(-\frac{i}{2\sigma^2}\int_{t_a}^{t_b}\mu^2(t)dt\right)\exp\left(-\frac{i}{4\kappa}\int_{t_a}^{t_b}f^2(t)dt\right)$$
$$\times\exp\left(-\frac{1}{2\sigma^2\sqrt{\kappa}\sigma}\int_{t_a}^{t_b}\int_{t_a}^{t}\sigma^4\left[f(t)-\frac{\mu'(t)}{\sigma^2}\right]\left[f(s)-\frac{\mu'(s)}{\sigma^2}\right]e^{-i\sqrt{\kappa}\sigma(t-s)}dsdt\right)$$

We then can finally expand the expression with regard to G_{mn} in powers of a and b and compare terms to get

$$G_{mn}(t_b,t_a) = \exp\left(-\frac{i}{2\sigma^2}\int_{t_a}^{t_b}\mu^2(t)dt\right)\exp\left(-\frac{i}{4\kappa}\int_{t_a}^{t_b}f^2(t)dt\right)$$
$$\times\frac{G_{00}}{\sqrt{m!n!}}\sum_{r=0}^{k}\frac{m!}{(m-r)!r!}\frac{n!}{(n-r)!r!}r!$$

$$\times\left(\frac{i}{\sqrt{2\sigma^2\sqrt{\kappa}\sigma}}\int_{t_a}^{t_b}\sigma^2\left[f(t)-\frac{\mu'(t)}{\sigma^2}\right]e^{-i\sqrt{\kappa}\sigma t}dt\right)^{n-r}\left(\frac{i}{\sqrt{2\sigma^2\sqrt{\kappa}\sigma}}\int_{t_a}^{t_b}\sigma^2\left[f(t)-\frac{\mu'(t)}{\sigma^2}\right]e^{i\sqrt{\kappa}\sigma t}dt\right)^{m-r}$$

where $k=\min(m,n)$ and

$$G_{00}=\exp\left(-\frac{i}{2\sigma^2}\int_{t_a}^{t_b}\mu^2(t)dt\right)\exp\left(-\frac{i}{4\kappa}\int_{t_a}^{t_b}f^2(t)dt\right)$$

$$\times\exp\left(-\frac{1}{2\sigma^2\sqrt{\kappa}\sigma}\int_{t_a}^{t_b}\int_{t_a}^{t}\sigma^4\left[f(t)-\frac{\mu'(t)}{\sigma^2}\right]\left[f(s)-\frac{\mu'(s)}{\sigma^2}\right]e^{-i\sqrt{\kappa}\sigma(t-s)}dsdt\right).$$

Hence, the eigenvalue-eigenfunction representation of kernel $K(x_b,t_b;x_a,t_a)$ with time-varying inflation $-\mu(t)$ without reinjections of firms is expressed as

$$K(x_b,t_b;x_a,t_a)=\sum_m\sum_n G_{mn}(t_b,t_a)\phi_m(x_b)\phi_n(x_a)e^{-\frac{i}{\sigma^2}\lambda_m t_b},$$

where $G_{mn}(t_b,t_a)$ is given above and

$$\phi_m(x_b)=\frac{1}{(2^m m!)^{1/2}}\left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/4}H_m\left(\sqrt{\frac{\sqrt{\kappa}\sigma}{\sigma^2}}x_b\right)e^{-\left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)x_b^2}.$$

$$\phi_n(x_a)=\frac{1}{(2^n n!)^{1/2}}\left(\frac{\sqrt{\kappa}\sigma}{\pi\sigma^2}\right)^{1/4}H_n\left(\sqrt{\frac{\sqrt{\kappa}\sigma}{\sigma^2}}x_a\right)e^{-\left(\frac{\sqrt{\kappa}\sigma}{2\sigma^2}\right)x_a^2}$$

$$\lambda_m=\sqrt{\kappa}\sigma^3\left(m+\frac{1}{2}\right).$$

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