Idiosyncratic Risk and the Equity Premium*

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Abstract

This paper aims to further our understanding of the effects of idiosyncratic risk on the equity premium. We consider different classes of preferences and different co-variations between the idiosyncratic shock's volatility and the economy's aggregate income. For short-lived assets, we offer a complete characterization of the effect, relying on the cross-moments of different derivatives of the utility function and the aggregate income of the economy. For long-lived assets, a full characterization is elusive, but we present sufficient conditions for the reversal of the effects found for short-lived assets.
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In an economy with aggregate risk, the equity premium is the difference between the expected return of a dollar invested in an asset bearing the same risk as the whole economy and the risk-free interest rate. Equivalently, the equity premium measures how much more expensive a risk-less asset that pays the expected return of the economy is, relative to the price of the risky return itself. This variable gained notoriety after [18] observed a significant difference between the empirical equity premium of the U.S. economy and its theoretical counterpart: the equity premium puzzle is the observation that standard macroeconomic models with homogeneous agents (both ex-ante and ex-post) fail to explain the equity premia typically observed in the data.[1]

Shortly after, [15] presented a setting in which ex-post heterogeneity affects the equity premium predicted by homogeneous agent models. Mankiw first observed that, in the presence of uninsurable idiosyncratic risk, the way in which [18] modeled the equity premium would be a miss-specification, except under the assumption that all the agents in the economy have quadratic preferences.[2] This insight was extended by [23], who showed that higher-order derivatives of the agents' Bernoulli utility function matter for the determination of the effects of idiosyncratic risk on asset prices and premia.

Later on, [4] again observed that failing to account for ex-post heterogeneity is akin to miss-estimating fundamentals such as the discount factor or the risk aversion coefficient. Indeed,

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1 See also [10] and [17].

2 Under these preferences, the agents don't demand savings for precautionary reasons, so the presence of idiosyncratic shocks does not affect the equilibrium prices of assets.

they observed theoretical equity premium is higher, in the case of CRRA preferences, when there is counter-cyclically heteroskedastic idiosyncratic risk.[3]

Unfortunately, the magnitude and even the direction of the effects of ex-post heterogeneity on the equity premium seem to depend on details and assumptions of the model used to predict it theoretically. For instance, for the same class of preferences as [4], [22] showed that in an OLG economy, the effect of idiosyncratic shocks on the equity premium is significantly smaller than in the case considered by [4].

Moreover, [13] argued that the economy's stochastic discount factor is independent of the volatility of idiosyncratic shocks when the agents have CRRA preferences and the ratio of the stochastic to the aggregate shocks is homoskedastic. Through this mechanism, [12] showed that the result of [4] does not hold true when the agents in the economy have CRRA preferences and the distribution of idiosyncratic risk follows a particular form of pro-cyclical heteroskedasticity in a two-period economy. Under these assumptions, the equity premium is not affected by the presence of idiosyncratic risk.[4]

This paper aims to further our understanding of the effect of idiosyncratic risk on the equity premium. We consider different classes of preferences and different co-variations between the idiosyncratic shocks' variance and the economy's aggregate income. For short-lived assets, such as those considered in [12], we offer a complete characterization of the effect, relying on the cross-moments of different derivatives of the utility function and the aggregate income of the economy. For long-lived assets, such as those in [22], a full characterization is elusive, but we present sufficient conditions for the reversal of the effect found by [4].

We also study the effects of higher-order moments of the distribution of idiosyncratic risk. This exercise is motivated by the theoretical results of [16] and the empirical work of [8], which highlight the importance of higher moments of the distribution of idiosyncratic shocks, in particular its negative skewness and high kurtosis.[5] Our main results rely on a Taylor expansion of the representative investor's marginal utility function and empirical papers also apply this technique. For instance, [2] derives bounds fo the market excess return by taking a Taylor expansion on the stochastic discount factors.

1 A Two-Period Homogeneous Economy

Consider a society consisting of a unit mass of ex-ante identical individuals who live over two periods. In the present each individual's wealth is the constant $\bar{w} > 0$. Their future wealth is the non-degenerate random variable W, whose support is a subset $\mathcal{W}$ of $\mathbb{R}_{++}$.

The preferences of each individual over present consumption, c, and future risky consumption, C, are of the Selden type, namely represented by the function

$$v(c) + \beta v(u^{-1}(\mathbb{E}[u(C)])), \tag{1}$$

where $v : \mathbb{R}_+ \to \mathbb{R}$ and $\beta > 0$ capture the agent's preference for inter-temporal consumption smoothing and her impatience, while $u : \mathbb{R}_+ \to \mathbb{R}$ is the agent's Bernoulli utility index, which models her attitude towards risk.[6] Assume that u is $\mathbf{C}^1(\mathbb{R}_{++})$, strictly increasing and concave,

[3] See also the empirical results in [3].

[4] See also [24] and [19].

[5] See also [11] and [1]. [20] and [9] study how the third and fourth moments of the distribution of risk affect investment decisions.

[6] See [21].

whereas u is $\mathbf{C}^3(\mathbb{R}_{++})$, strictly increasing and strictly concave and has non-negative third derivative.

Only one asset can be traded: the asset that pays W in the second period. We interpret this asset as the "equity" of the economy.

1.1 Benchmark: only aggregate risk

In the absence of any other shocks, the present and future consumption of an individual in this economy are, respectively, $c = \bar{w} - q \cdot y$ and $C = W + W \cdot y$, where q denotes the price of the asset and y is the quantity of the asset demanded by the individual. The portfolio problem of each agent is, hence,

$$\max_y \left\{ v(\bar{w} - q \cdot y) + \beta v(u^{-1}(\mathbb{E}[u(W + W \cdot y)])) \right\}.$$

Since all agents are identical, only a no-trade equilibrium is possible and

$$q = \frac{\beta v'(u^{-1}(\mathbb{E}[u(W)]))}{v'(\bar{w})} \frac{\mathbb{E}[u'(W) \cdot W]}{u'(u^{-1}(\mathbb{E}[u(W)]))}. \tag{2}$$

If we define the function $m : \mathbb{R}_{++} \to \mathbb{R}$, as

$$m(w) = \frac{\beta v'(u^{-1}(\mathbb{E}[u(W)]))}{v'(\bar{w})} \cdot \frac{\mathbb{E}[u'(w)]}{u'(u^{-1}(\mathbb{E}[u(W)]))},$$

this economy's *stochastic discount factor* is the random variable $m(W)$ and we can re-write Eq. (2) as $q = \mathbb{E}[m(W) \cdot W]$.

Using the same kernel for pricing other income flows, note that if the agents could also trade a risk-less asset with payoff $\mathbb{E}(W)$, its price would equal

$$\mathbb{E}[m(W) \cdot \mathbb{E}(W)] = \mathbb{E}[m(W)] \cdot \mathbb{E}(W) = \frac{\beta v'(u^{-1}(\mathbb{E}[u(W)]))}{v'(\bar{w})} \cdot \frac{\mathbb{E}[u'(W)] \cdot \mathbb{E}(W)}{u'(u^{-1}(\mathbb{E}[u(W)]))}.$$

The *equity premium*, in the absence of any other risk, is

$$\bar{p} = \frac{\mathbb{E}[m(W)] \cdot \mathbb{E}(W)}{\mathbb{E}[m(W) \cdot W]} - 1 = \frac{\mathbb{E}\left[u'(W)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u'(W) \cdot W\right]} - 1 = -\frac{\mathbb{C}\text{ov}[u'(W), W]}{\mathbb{E}[u'(W) \cdot W]}, \tag{3}$$

where $\mathbb{C}$ov is the *covariance* operator.[7]

That the constant

$$\frac{\beta v'(u^{-1}(\mathbb{E}[u(W)]))}{v'(\bar{w})} \cdot \frac{1}{u'(u^{-1}(\mathbb{E}[u(W)]))}$$

cancels out in the computation this relative price simply says that the equity premium depends on the agents' attitude towards risk but not on their impatience or attitude towards intertemporal smoothing. While these two latter considerations affect the prices of both the risky and the risk-less assets, their relative price depends *only* on the individual's attitude towards risk.[8]

[7] In the asset pricing literature, the risk free rate r_f is defined by $1 + r_f = 1/\mathbb{E}[m(W)]$, while the realized return of the market portfolio is the random variable defined by $1 + R_m = W/q = W/\mathbb{E}[m(W) \cdot W]$. The empirical equity premium equals

$$\mathbb{E}(R_m - r_f) = \frac{\mathbb{E}(W)}{\mathbb{E}[m(W) \cdot W]} - \frac{1}{\mathbb{E}[m(W)]} = \frac{1}{\mathbb{E}[m(W)]} \cdot \left\{ \frac{\mathbb{E}[m(W)] \cdot \mathbb{E}(W)}{\mathbb{E}[m(W) \cdot W]} - 1 \right\},$$

which implies that $\bar{p} = \mathbb{E}(R_m - r_f)/(1 + r_f)$. Our definition of the equity premium is, thus, the empirical equity premium discounted at the risk free rate.

[8] The use of Selden preferences generalizes the more traditional Expected Utility approach: if one lets $v = u$,

1.2 Idiosyncratic risk

While all the agents in the economy are ex-ante identical, we want to consider the effects of ex-post heterogeneity. To model this, let random variable S, with $\mathbb{E}(S \mid W) = 0$, be each agent's uninsurable, future idiosyncratic risk. When holding y units of the risky asset, an agent's future consumption is $C = W + S + W \cdot y$ and the equity premium is

$$p = \frac{\mathbb{E}\left[m(C)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[m(C) \cdot W\right]} - 1 = \frac{\mathbb{E}\left[u'(C)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u'(C) \cdot W\right]} - 1. \tag{4}$$

At equilibrium, iterating expectations, this is

$$p = \frac{\mathbb{E}\left\{\mathbb{E}\left[u'(W+S) \mid W\right]\right\} \cdot \mathbb{E}(W)}{\mathbb{E}\left\{\mathbb{E}\left[u'(W+S) \mid W\right] \cdot W\right\}} - 1. \tag{5}$$

Note from Eq. (5) that if the economy displays idiosyncratic risk, using Eq. (3) instead of Eq. (4) misspecifies the equity premium, as it amounts to assuming that

$$\mathbb{E}\left[u'(W+S) \mid W\right] = u'\left(\mathbb{E}(W+S \mid W)\right),$$

which in general requires that the Bernoulli function be quadratic.[9]

From now on, we assume that the individuals display strict prudence, namely that $u''' > 0$.

1.3 Idiosyncratic risk and the equity premium

Using the expansion

$$u'(w+s) \approx u'(w) + u''(w) \cdot s + \tfrac{1}{2} \cdot u'''(w) \cdot s^2, \tag{6}$$

we get that

$$\mathbb{E}\left[u'(W+S) \mid W\right] \approx u'(W) + \tfrac{1}{2} \cdot u'''(W) \cdot \mathbb{V}(S \mid W),$$

where $\mathbb{V}$ is the variance operator. This allows us to approximate Eq. (5) by

$$\hat{p} = \frac{\mathbb{E}\left[u'(W) + \frac{1}{2} \cdot u'''(W) \cdot \mathbb{V}(S \mid W)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left\{\left[u'(W) + \frac{1}{2} \cdot u'''(W) \cdot \mathbb{V}(S \mid W)\right] \cdot W\right\}} - 1. \tag{7}$$

In order to perform comparative statics, we parameterize the conditional variance of the idiosyncratic shock by assuming that $\mathbb{V}(S \mid W = w) = \sigma^2 w^\eta$ almost surely, for constants $\sigma > 0$ and η.

The following result is the most basic one in the paper. Still, we prove it in detail, as the proofs of more involved results will resemble this argument.[10]

THEOREM 1. *The equity premium $\hat{p}$ ranges monotonically from $\bar{p}$, when $\sigma = 0$, to*

$$\lim_{\sigma \to \infty} \hat{p} = \frac{\mathbb{E}\left[u'''(W) \cdot W^\eta\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u'''(W) \cdot W^{\eta+1}\right]} - 1.$$

Moreover, the following four statements are equivalent:

Eq. (1) becomes $u(c) + \beta\mathbb{E}[u(C)]$. In fact, a similar argument to the previous observation yields that our results are valid in the case where the individual's preferences are represented by the function $v(c) + \beta\mathbb{E}[u(C)]$.

[9] This observation is Proposition 1 in [15] and part of the motivation for [23].

[10] The theorem that follows continues to hold true for any parameterization of the conditional variance of the form $\mathbb{V}(S \mid W = w) = \sigma^2 \cdot \eta(w)$, for any function $\eta : \mathcal{W} \to \mathbb{R}_{++}$. The subsequent results require the specific parameterization we are using.

(a) $\hat{p} \gtreqqless \bar{p}$;

(b) $\dfrac{\partial \hat{p}}{\partial \sigma} \gtreqqless 0$;

(c) $\dfrac{\mathbb{E}\left[u'''(W) \cdot W^{\eta}\right]}{\mathbb{E}\left[u'''(W) \cdot W^{\eta+1}\right]} \gtreqqless \dfrac{\mathbb{E}\left[u'(W)\right]}{\mathbb{E}\left[u'(W) \cdot W\right]}$; *and*

(d) $\dfrac{\mathbb{C}\mathrm{ov}[u'''(W) \cdot W^{\eta}, W]}{\mathbb{E}[u'''(W) \cdot W^{\eta+1}]} \lesseqqgtr \dfrac{\mathbb{C}\mathrm{ov}[u'(W), W]}{\mathbb{E}[u'(W) \cdot W]}$.

Proof. Under the assumed functional form of $\mathbb{V}(S \mid W)$, Eq. (7) rewrites as

$$\hat{p} = \frac{\mathbb{E}\left[u'(W)\right] \cdot \mathbb{E}(W) + \frac{1}{2} \cdot \mathbb{E}\left[u'''(W) \cdot W^{\eta}\right] \cdot \mathbb{E}(W) \cdot \sigma^2}{\mathbb{E}\left[u'(W) \cdot W\right] + \frac{1}{2} \cdot \mathbb{E}\left[u'''(W) \cdot W^{\eta+1}\right] \cdot \sigma^2} - 1.$$

The two limits follow, thus, by direct computation.

The equivalence between (a) and (b) is straightforward. To see that (b) and (c) are equivalent, it simplifies our notation if we write

$$\hat{p} = \frac{N + \frac{1}{2} \cdot \mathbb{E}\left[u'''(W) \cdot W^{\eta}\right] \cdot \mathbb{E}(W) \cdot \sigma^2}{D + \frac{1}{2} \cdot \mathbb{E}\left[u'''(W) \cdot W^{\eta+1}\right] \cdot \sigma^2} - 1,$$

where $N = \mathbb{E}\left[u'(W)\right] \cdot \mathbb{E}(W)$ and $D = \mathbb{E}\left[u'(W) \cdot W\right]$ are, respectively, the numerator and the denominator in the definition of $\bar{p}$, as per Eq. (3). By direct computation, and since $\sigma > 0$, note that $\hat{p}$ is increasing, constant or decreasing in σ depending on whether

$$D \cdot \mathbb{E}(W) \cdot \mathbb{E}[u'''(W) \cdot W^{\eta}] \gtreqqless N \cdot \mathbb{E}(W) \cdot \mathbb{E}\left[u'''(W) \cdot W^{\eta+1}\right].$$

By monotonicity and strict prudence, and since W takes only positive values, $D > 0$, $\mathbb{E}(W) > 0$, and $\mathbb{E}\left[u'''(W) \cdot W^{\eta+1}\right] > 0$. Thus, we can rewrite this expression as

$$\frac{\mathbb{E}[u'''(W) \cdot W^{\eta}]}{\mathbb{E}\left[u'''(W) \cdot W^{\eta+1}\right]} \gtreqqless \frac{N}{D},$$

namely statement (c).

Finally, to see that (c) and (d) are equivalent, note that

$$\begin{aligned}
&\frac{\mathbb{E}\left[u'''(W) \cdot W^{\eta}\right]}{\mathbb{E}\left[u'''(W) \cdot W^{\eta+1}\right]} \gtreqqless \frac{\mathbb{E}\left[u'(W)\right]}{\mathbb{E}\left[u'(W) \cdot W\right]} \\
\Leftrightarrow\ &\frac{\mathbb{E}\left[u'''(W) \cdot W^{\eta}\right] \cdot E(W)}{\mathbb{E}\left[u'''(W) \cdot W^{\eta+1}\right]} - 1 \gtreqqless \frac{\mathbb{E}\left[u'(W)\right] \cdot E(W)}{\mathbb{E}\left[u'(W) \cdot W\right]} - 1 \\
\Leftrightarrow\ &-\frac{\mathbb{C}\mathrm{ov}\left[u'''(W) \cdot W^{\eta}, W\right]}{\mathbb{E}\left[u'''(W) \cdot W^{\eta+1}\right]} \gtreqqless -\frac{\mathbb{C}\mathrm{ov}\left[u'(W), W\right]}{\mathbb{E}\left[u'(W) \cdot W\right]}
\end{aligned}$$

□

Later on, it will be useful write condition (c) more concisely as

$$\frac{\mathbb{E}\left[u'''(W) \cdot W^{\eta}\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u'''(W) \cdot W^{\eta+1}\right]} - 1 \gtreqqless \bar{p}.$$

Also, note that the denominators on both sides of the expression in condition (d) are positive, and risk aversion implies that the numerator on its left-hand side is negative, so the ratio on the left-hand side is negative. None of our assumptions so far pins down the sign of the numerator on the right-hand side, though.[11]

[11] If the covariance between the economy's aggregate income and the third derivative of the Bernoulli function were positive, then an increase in the variance of the idiosyncratic shock would *decrease* the equity premium.

2 Two Important Examples

To obtain concrete results, we now consider two canonical classes of Bernoulli functions.

2.1 CARA Preferences

In order to study the class of functions that display constant absolute risk aversion, in this section we assume that the Bernoulli function is exponential, namely that $u(w) = -e^{-\alpha w}$ for some constant $\alpha > 0$.

Theorem 2. *Suppose that u displays CARA. Whether the equity premium is larger, equal, or smaller in the presence of idiosyncratic risk depends on whether this risk is counter-, a-, or pro-cyclical. That is,*

$$\hat{p} \gtreqless \bar{p} \Leftrightarrow \eta \lesseqgtr 0.$$

Proof. We know from Theorem 1, by direct computation, that under under this functional form

$$\hat{p} \gtreqless \bar{p} \Leftrightarrow \frac{\mathbb{E}(\alpha^3 e^{-\alpha W} \cdot W^\eta)}{\mathbb{E}(\alpha^3 e^{-\alpha W} \cdot W^{\eta+1})} \gtreqless \frac{\mathbb{E}(\alpha e^{-\alpha W})}{\mathbb{E}(\alpha e^{-\alpha W} \cdot W)}.$$

Whether $\hat{p}$ is larger, equal, or smaller than $\bar{p}$ depends thus on the sign of

$$\mathbb{E}(e^{-\alpha W} \cdot W^\eta) \cdot \mathbb{E}(e^{-\alpha W} \cdot W) - \mathbb{E}(e^{-\alpha W}) \cdot \mathbb{E}(e^{-\alpha W} \cdot W^{\eta+1}).$$

If we let V be an (ancillary) random variable distributed identically to W and independent from it, we can rewrite the latter expression as $\mathbb{E}\left[e^{-\alpha(W+V)} \cdot W^\eta \cdot (V - W)\right]$, which is proportional, by a factor of $1/2 \Pr(V \neq W) > 0$, to

$$\mathbb{E}\left[e^{-\alpha(W+V)} \cdot W^\eta \cdot (V - W) \mid V > W\right] + \mathbb{E}\left[e^{-\alpha(W+V)} \cdot W^\eta \cdot (V - W) \mid V < W\right].$$

This expression is equivalent to

$$\mathbb{E}\left[e^{-\alpha(W+V)} \cdot W^\eta \cdot (V - W) \mid V > W\right] + \mathbb{E}\left[e^{-\alpha(V+W)} \cdot V^\eta \cdot (W - V) \mid W < V\right],$$

which, by direct computation, is

$$\mathbb{E}\left[e^{-\alpha(W+V)} \cdot (W^\eta - V^\eta) \cdot (V - W) \mid V > W\right].$$

This number is positive, null, or negative, depending on whether η is negative, null, or positive. $\square$

Note that if $\eta = 0$, the argument is pretty simple:

$$\hat{p} = \frac{\mathbb{E}(\alpha^3 e^{-\alpha W})}{\mathbb{E}(\alpha^3 e^{-\alpha W} \cdot W)} = \frac{\mathbb{E}(\alpha e^{-\alpha W})}{\mathbb{E}(\alpha e^{-\alpha W} \cdot W)} = \bar{p}.$$

2.2 CRRA preferences

We now focus on Bernoulli functions with constant relative risk aversion. The property that this class gives us is that the first derivative of the Bernoulli function is homogeneous, so we can write $u'(w) = u'(1) \cdot w^{-\rho}$ for some constant $\rho > 0$.

Theorem 3. *Suppose that u displays CRRA. Whether the equity premium is larger, the same, or smaller in the presence of idiosyncratic risk, depends on whether η is smaller, equal, or larger than 2. That is,*

$$\hat{p} \gtreqless \bar{p} \Leftrightarrow \eta \lesseqgtr 2.$$

Proof. Substituting the functional form of the conditional variance of S, we get, again by Theorem 1, that

$$\hat{p} \gtreqless \bar{p} \Leftrightarrow \frac{\mathbb{E}\left[\rho(1+\rho)\cdot W^{\eta-\rho-2}\right]}{\mathbb{E}\left[\rho(1+\rho)\cdot W^{\eta-\rho-1}\right]} \gtreqless \frac{\mathbb{E}(W^{-\rho})}{\mathbb{E}(W^{-\rho+1})}.$$

We thus need to show that $\eta < 2$ is necessary and sufficient for

$$\mathbb{E}(W^{\eta-\rho-2})\cdot\mathbb{E}(W^{-\rho+1}) > \mathbb{E}(W^{-\rho})\cdot\mathbb{E}(W^{\eta-\rho-1}).$$

Let us define random variable V as in the proof of Theorem 2. By direct computation, we need to argue that

$$\mathbb{E}\left[W^{-\rho}\cdot V^{-\rho+1}\cdot(W^{\eta-2}-V^{\eta-2})\right] > 0.$$

Using the same technique as in the proof of Theorem 2, the left-hand side of this expression is directly proportional to

$$\mathbb{E}\left[W^{-\rho}\cdot V^{-\rho}\cdot(V-W)\cdot(W^{\eta-2}-V^{\eta-2}) \mid W > V\right].$$

This expression is positive if, and only if, $\eta < 2$. □

As before, note that if $\mathbb{V}(S \mid W) = \sigma^2 W^2$, with $u'''(w) = \rho(\rho+1)u(1)\cdot w^{-(\rho+2)}$, we get

$$\hat{p} = \frac{\mathbb{E}\left\{\left[1+\rho(\rho+1)\frac{\sigma^2}{2}\right]\cdot W^{-\rho}\right\}\cdot\mathbb{E}(W)}{\mathbb{E}\left\{\left[1+\rho(\rho+1)\frac{\sigma^2}{2}\right]\cdot W^{-\rho+1}\right\}} - 1 = \bar{p}.$$

2.3 Risk aversion and the cyclicality of the volatility of idiosyncratic shocks

A comparison of the previous two theorems suggests a connection between the behavior of the coefficients of risk aversion, the behavior of the conditional variance of the idiosyncratic shocks, and the effect of the latter on the equity premium.

As is well known, the absolute risk aversion coefficient approximates the willingness to pay to insure against *additive* shocks of variance 2, and Theorem 2 states that when such willingness to pay is constant, the equity premium increases, remains, or decreases, depending on whether the volatility of the *absolute* idiosyncratic shock S is countercyclical, acyclical, or procyclical.

The relative risk aversion coefficient, on the other hand, approximates an agents' willingness to pay to insure against *multiplicative* shocks of variance 2. In the class of preferences of Theorem 3, this coefficient is constant, and we can re-write the conclusion of that theorem as saying that the presence of idiosyncratic increases, decreases, or leaves the risk premium unchanged depending on whether the volatility of the *relative* idiosyncratic shock S/W is countercyclical, acyclical, or procyclical.

To reiterate, in particular:

Corollary 1. *Suppose that the absolute idiosyncratic shock is homoskedastic, so that* $\mathbb{V}(S \mid W) = \sigma^2 > 0$ *almost surely. Then:*

(a) if the Bernoulli function exhibits CARA, $\hat{p}$ *does not depend on* σ*; while*

(b) if it displays CRRA, $\hat{p}$ *is increasing in* σ*.*

If, alternatively, the relative idiosyncratic shock is homoskedastic and $\mathbb{V}(S/w \mid W) = \sigma^2 > 0$ *almost surely, then:*

(c) if the Bernoulli function displays CARA, $\hat{p}$ *is decreasing in* σ*; while*

(d) if it is of CRRA, $\hat{p}$ *does not depend on* σ*.*

3 Skewness

One can see the equity premium defined in Eq. (7) as a correction to Eq. (3) that takes into account the effects of the variance of idiosyncratic risk *only*. [8], however, has noted the significance of the latter risk's negative skewness. This section addresses a further correction to the equity premium that takes into account this moment. We determine the effect of skewness on the premium, re-calculate the effect of the variance, and calculate how the skewness changes the magnitude of the latter effect.

Our previous analysis relied on the second-order expansion (6). From now on, we refer to the premium resulting for this expansion, namely Eq. (7), as $\hat{p}_2$. The third-order expansion

$$u'(w+s) \approx u'(w) + u''(w)\cdot s + \tfrac{1}{2}\cdot u'''(w)\cdot s^2 + \tfrac{1}{6}\cdot u^{[4]}(w)\cdot s^3$$

yields

$$\mathbb{E}\left[u'(W+S)\mid W\right] \approx u'(W) + \tfrac{1}{2}\cdot u'''(W)\cdot \mathbb{V}(S\mid W) + \tfrac{1}{6}\cdot u^{[4]}(W)\cdot \mathbb{E}(S^3\mid W).$$

We will assume that the distribution of idiosyncratic is the constant

$$\gamma = \operatorname{Skew}(S\mid W) = \frac{\mathbb{E}(S^3\mid W)}{\mathbb{V}(S\mid W)^{3/2}} \leq 0,$$

and maintain the assumption that $\mathbb{V}(S \mid W) = \sigma^2 W^\eta$. In this setting we define the premium

$$\hat{p}_3 = \frac{\mathbb{E}\left[u'(W) + \frac{\sigma^2}{2}\cdot u'''(W)\cdot W^\eta + \frac{\gamma\sigma^3}{6}\cdot u^{[4]}(W)\cdot W^{3\eta/2}\right]\cdot \mathbb{E}(W)}{\mathbb{E}\left\{\left[u'(W) + \frac{\sigma^2}{2}\cdot u'''(W)\cdot W^\eta + \frac{\gamma\sigma^3}{6}\cdot u^{[4]}(W)\cdot W^{3\eta/2}\right]\cdot W\right\}} - 1,$$

which explicitly corrects for the effect of the skewness and re-defines the effect of the variance on the equity premium.

In order to ensure positive asset prices, we further assume that the Bernoulli function is *temperate*, or risk-averse of order 4, namely that $u^{[4]} \leq 0$.[12] Our first result is on the direct effect of the skewness on the premium:

Theorem 4. *The equity premium $\hat{p}_3$ ranges monotonically between $\hat{p}_2$, when $\gamma = 0$, and*

$$\lim_{\gamma\to-\infty} \hat{p}_3 = \frac{\mathbb{E}\left[u^{[4]}(W)\cdot W^{3\eta/2}\right]\cdot \mathbb{E}(W)}{\mathbb{E}\left[u^{[4]}(W)\cdot W^{3\eta/2}\cdot W\right]} - 1.$$

Moreover, the following three statements are equivalent:

(a) $\hat{p}_3 \gtreqless \hat{p}_2$;

(b) $\dfrac{\partial \hat{p}_3}{\partial \gamma} \gtreqless 0$; *and*

(c) $\dfrac{\mathbb{E}\left[u^{[4]}(W)\cdot W^{3\eta/2}\right]\cdot \mathbb{E}(W)}{\mathbb{E}\left[u^{[4]}(W)\cdot W^{3\eta/2}\cdot W\right]} - 1 \lesseqgtr \hat{p}_2$.

The proof of this result is very similar to the argument for Theorem 1, so we defer it, along with all the other proofs in the paper, to an appendix.[13]

Our next result re-calculates the effect of the variance, given the skewness:

[12] See [6].

[13] A condition analogous to (d) in Theorem 1 can be derived, but it is rather cumbersome and uninformative.

THEOREM 5. *The equity premium $\hat{p}_3$ ranges between $\bar{p}$, when $\sigma = 0$, and*

$$\lim_{\sigma\to\infty} \hat{p}_3 = \frac{\mathbb{E}\left[u^{[4]}(W)\cdot W^{3\eta/2}\right]\cdot \mathbb{E}(W)}{\mathbb{E}\left[u^{[4]}(W)\cdot W^{3\eta/2}\cdot W\right]} - 1.$$

Moreover, $\hat{p}_3 \geq \bar{p}$ and $\partial\hat{p}_3/\partial\sigma \geq 0$ if

$$\frac{\mathbb{E}[u^{[4]}(W)\cdot W^{3\eta/2}]}{\mathbb{E}\left[u^{[4]}(W)\cdot W^{3\eta/2+1}\right]} \geq \frac{\mathbb{E}[u'''(W)\cdot W^{\eta}]}{\mathbb{E}\left[u'''(W)\cdot W^{\eta+1}\right]} \geq \frac{\mathbb{E}[u'(W)]}{\mathbb{E}[u'(W)\cdot W]}. \tag{8}$$

If both *of these inequalities fail, then $\hat{p}_3 \leq \bar{p}$ and $\partial\hat{p}_3/\partial\sigma \leq 0$.*

Finally, we compute how the skewness of the idiosyncratic shock affects the way in which its variance impacts the premium.

THEOREM 6. *Suppose first that Eq.* (8) *holds. Then,*

(a) $\dfrac{\partial^2\hat{p}_3}{\partial\gamma\partial\sigma^2} \geq 0$ *if*

$$\gamma \leq \min\left\{\frac{\mathbb{E}[u'(W)\cdot W] + \frac{\sigma^2}{2}\cdot\mathbb{E}\left[u'''(W)\cdot W^{\eta+1}\right]}{\frac{\sigma^3}{6}\cdot\mathbb{E}\left[u^{[4]}(W)\cdot W^{3\eta/2+1}\right]}, \frac{3\mathbb{E}\left[u'''(W)\cdot W^{\eta+1}\right]}{\sigma^4\cdot\mathbb{E}\left[u^{[4]}(W)\cdot W^{3\eta/2+1}\right]}\right\}; \tag{9}$$

(b) $\dfrac{\partial^2\hat{p}_3}{\partial\gamma\partial\sigma^2} \leq 0$ *if*

$$\gamma \geq \max\left\{\frac{\mathbb{E}[u'(W)\cdot W] + \frac{\sigma^2}{2}\cdot\mathbb{E}\left[u'''(W)\cdot W^{\eta+1}\right]}{\frac{\sigma^3}{6}\cdot\mathbb{E}\left[u^{[4]}(W)\cdot W^{3\eta/2+1}\right]}, \frac{3\mathbb{E}\left[u'''(W)\cdot W^{\eta+1}\right]}{\sigma^4\cdot\mathbb{E}\left[u^{[4]}(W)\cdot W^{3\eta/2+1}\right]}\right\}. \tag{10}$$

On the other hand, if both *inequalities in Eq.* (8)*, then the condition in Eq.* (9) *suffices for $\partial^2\hat{p}_3/\partial\gamma\partial\sigma^2 \leq 0$, while the condition in Eq.* (10) *implies that $\partial^2\hat{p}_3/\partial\gamma\partial\sigma^2 \geq 0$.*

For the two canonical families of Bernoulli functions studied in Section 2, these theorems allow us to determine the effects unambiguously. For the sake of concreteness and to maintain simpler expressions, we consider only the cases when the absolute and relative idiosyncratic shock is homoskedastic.

Since the exponential Bernoulli function, of Subsection 2.1is risk averse of order 4, we can apply the previous theorems:

THEOREM 7. *Suppose that u displays CARA.*

(a) If the absolute idiosyncratic shock is homoskedastic, namely if $\eta = 0$, then $\hat{p}_3$ does neither on the variance σ nor on the skewness γ. In fact, $\hat{p}_3 = \hat{p}_2 = \bar{p}$.

(b) If, on the other hand, the relative idiosyncratic shock is homoskedastic, so $\eta = 2$, then $\hat{p}_3$ is decreasing in σ and increasing in γ, and $\hat{p}_3 \leq \hat{p}_2 < \bar{p}$. Also,

$$\frac{\partial^2\hat{p}_3}{\partial\gamma\partial\sigma^2} \geq (\leq)\, 0$$

for

$$\gamma \geq \max\,(\leq \min)\left\{-\frac{\mathbb{E}\left[e^{-\alpha W}\cdot W\cdot(1+\alpha^2\sigma^2/2W^2)\right]}{\alpha^3\sigma^3\cdot E(e^{-\alpha W}\cdot W^4)}, -\frac{3\cdot\mathbb{E}(e^{-\alpha W}\cdot W^3)}{\alpha\sigma^4\cdot\mathbb{E}(e^{-\alpha W}\cdot W^4)}\right\}.$$

As for the CRRA preferences of Subsection 2.2, since they too are risk-averse of degree 4:

THEOREM 8. *Suppose that u displays CRRA.*

(a) If the absolute idiosyncratic shock is homoskedastic, so $\eta = 0$, then $\hat{p}_3$ is increasing in σ and decreasing in γ. Moreover, $\hat{p}_3 \geq \hat{p}_2 > \bar{p}$ and

$$\frac{\partial^2 \hat{p}_3}{\partial\gamma\partial\sigma^2} \geq (\leq)\, 0$$

whenever

$$\gamma \leq \min\,(\geq \max) \left\{ -\frac{6 \cdot [\mathbb{E}(W^{-\rho+1}) + \sigma^2/2 \cdot (\rho+1) \cdot \mathbb{E}(W^{-\rho-1})]}{\sigma^3 \cdot (\rho+1) \cdot (\rho+2) \cdot \mathbb{E}(W^{-\rho-2})}, -\frac{3 \cdot \mathbb{E}(W^{-\rho-1})}{\sigma^4 \cdot (\rho+2)\mathbb{E}(W^{-\rho-2})} \right\}.$$

(b) If, on the other hand, the relative idiosyncratic shock is homoskedastic, namely if $\eta = 2$, then $\hat{p}_3$ depends nether on the variance σ nor on the skewness γ and $\hat{p}_3 = \hat{p}_2 = \bar{p}$.

4 Higher-order Moments

The effect of higher-order moments of the distribution of idiosyncratic shocks on the equity premium critically depends on the agents' higher-order risk aversion.[14] In this section, we provide a full characterization using the n^{th}-order expansion on the distribution of the marginal utility.

Suppose that the i^{th} standardized moment of the conditional distribution of S is the constant $\mu_i < \infty$ almost surely.[15] The n^{th}-order approximation to the marginal utility is, analogously to Eq. (6),

$$u'(w+s) \approx \sum_{i=1}^{n+1} \tfrac{1}{(i-1)!} \cdot u^{[i]}(w) \cdot s^{i-1}.$$

This expression yields a further correction to the equity premium,

$$\hat{p}_n = \frac{\mathbb{E}\left[\sum_{i=1}^{n+1} \frac{1}{(i-1)!} \cdot u^{[k]}(W) \cdot \mathbb{V}(S \mid W)^{\frac{i-1}{2}} \cdot \mu_{i-1}\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[\sum_{i=1}^{n+1} \frac{1}{(i-1)!} \cdot u^{[k]}(W) \cdot \mathbb{V}(S \mid W)^{\frac{i-1}{2}} \cdot \mu_{i-1} \cdot W\right]} - 1, \tag{11}$$

which we can use to study the effects of the higher-order moments in the same way we used Eq. (7) to study the effect of its variance.

Following [5], we will say that the Bernoulli function u is *risk-averse of order n* if $(-1)^i \cdot u^{[i]} < 0$ for all $i = 1, \dots, n$.[16] This property strengthens the usual hypothesis that u was strictly non-satiated, strictly risk-averse, and strictly prudent.

Our next result characterizes the effect of each standardized moment μ_n on $\hat{p}_n$ and allows to order the different measures of the premium. As before, we adopt the parameterization $V(S \mid W) = \sigma^2 W^\eta$, although this makes no significant difference in the following result.

[14] [6] show that the sign of the n^{th}-order derivative of the utility function characterizes agent's n^{th} order risk attitude. In addition, [14] extends the result to multiplicative risks.

[15] That is, that $E(S^i \mid W) = \mathbb{V}(S \mid W)^{i/2} \cdot \mu_i$, W-a.s., for all $i \geq 2$. We maintain the assumption that $\mu_1 = 0$, and adopt the convention that $\mu_0 = 1$.

[16] Risk aversion of order 4 is often referred to as *temperance*, whereas aversion of order 5 is called *edginess*.

Theorem 9. *Let $n \geq 3$, and suppose $(-1)^i \cdot \mu_i \geq 0$ for all $i = 1, \ldots, n$. If the Bernoulli function is risk-averse of order $n+1$, then the equity premium $\hat{p}_n$ ranges monotonically between $\lim_{\mu_n \to 0} \hat{p}_n = \hat{p}_{n-1}$ and*

$$\lim_{\mu_n \to (-1)^{n+1}\infty} \hat{p}_n = \frac{\mathbb{E}\left[u^{[n+1]}(W) \cdot W^{n\eta/2}\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u^{[n+1]}(W) \cdot W^{n\eta/2} \cdot W\right]} - 1.$$

Moreover, the following three statements are equivalent:

(a) $\hat{p}_n \gtreqless \hat{p}_{n-1}$*;*

(b) $\dfrac{\partial \hat{p}_n}{\partial \mu_n} \gtreqless 0$*; and*

(c) $(-1)^n \cdot \left\{ \dfrac{\mathbb{E}\left[u^{[n+1]}(W) \cdot W^{n\eta/2}\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u^{[n+1]}(W) \cdot W^{n\eta/2} \cdot W\right]} - 1 \right\} \gtreqless (-1)^n \cdot \hat{p}_{n-1}$ *.*

The theorem is just the first step in the determination of the effects of higher-order moments. For example, let us continue to denote the skewness of the conditional distribution of S by the constant $\gamma = \mu_3 \leq 0$, and let its kurtosis be the constant $\kappa = \mu_4 > 0$.

In the case of an CARA Bernoulli function:

(a) If the absolute idiosyncratic shock is homoskedastic, namely if $\eta = 0$, then $\hat{p}_3$ does not depend on the kurtosis κ. In fact, $\hat{p}_4 = \hat{p}_3 = \hat{p}_2 = \bar{p}$.

(b) If, on the other hand, the relative idiosyncratic shock is homoskedastic, so $\eta = 2$, then $\hat{p}_4$ is decreasing in κ and $\hat{p}_4 < \hat{p}_3 \leq \hat{p}_2 < \bar{p}$.

If, on the other hand, the function is of CRRA, then:

(a) If the absolute idiosyncratic shock is homoskedastic, so $\eta = 0$, then $\hat{p}_4$ is increasing in κ. Moreover, $\hat{p}_4 > \hat{p}_3 \geq \hat{p}_2 > \bar{p}$.

(b) If, on the other hand, the relative idiosyncratic shock is homoskedastic, namely if $\eta = 2$, then $\hat{p}_4$ does not depend on the kurtosis κ and $\hat{p}_4 = \hat{p}_3 = \hat{p}_2 = \bar{p}$.

5 Long-Lived Assets

Our analysis so far has assumed that the equity of the economy only pays dividend. If, more realistically, the capital of the economy is long-lived, we must adjust our analysis to take into account the re-sale value of the asset as part of its future return. For a specific application, consider the case of a stationary overlapping generations economy where individuals live for two periods and the only asset in the economy pays W, i.i.d., every period.

In every period, suppose that there and a unit mass of young individuals each of whom has an endowment $\bar{w}$, and a unit mass of old individuals who each own a unit of an asset whose dividend is the random variable W we studied before. If the asset is long-lived, the old agents get to consume both the dividend W and the price q of the asset. The young generation, in turn, pays q per unit of the asset, anticipating a payoff of $W + q$ one period later.

For reasons that will be clear below, we need to consider a simpler class of preferences to study this problem. In what follows, we assume that an agent that consumes c when young and C when old has lifetime utility $c + \mathbb{E}[u(C)]$, with a Bernoulli function u that displays risk aversion of order 3.

5.1 *Benchmark: only aggregate risk*

In the absence of any other risk, the problem of the young generation is

$$\max_y \{\bar{w} - q \cdot y + \mathbb{E}[u((W+q) \cdot y)]\},$$

where y represents, as before, the agent's demand for equity. The first-order condition of this problem is that

$$q = \mathbb{E}[u'((W+q) \cdot y) \cdot (W+q)],$$

while market clearing requires that $y = 1$, so q is the solution to the equation

$$q = \mathbb{E}[u'(W+q) \cdot (W+q)].$$

By the same arguments as before, a risk-less asset *with the same expected payoff*[17] should be priced at

$$\mathbb{E}[u'(W+q)] \cdot [\mathbb{E}(W) + q],$$

so the relative price of the risk-less asset to the risky asset (minus 1) is again the *equity premium*:

$$\bar{p} = \frac{\mathbb{E}[u'(W+q)] \cdot [\mathbb{E}(W) + q]}{\mathbb{E}[u'(W+q) \cdot (W+q)]} - 1. \tag{12}$$

5.2 *Idiosyncratic risk*

If the old generation faces idiosyncratic risk S, the premium is

$$p = \frac{\mathbb{E}[u'(W+q+S)] \cdot [\mathbb{E}(W) + q]}{\mathbb{E}[u'(W+q+S) \cdot (W+q)]} - 1, \tag{13}$$

and using Eq. (12) amounts to assuming that

$$\mathbb{E}[u'(W+q+S) \mid W] = u'(\mathbb{E}[W+q+S \mid W]).$$

In the same vein as Eq. (6), the quadratic expansion

$$u'(w+q+s) \approx u'(w+q) + u''(w+q) \cdot s + \tfrac{1}{2} \cdot u'''(w+q) \cdot s^2$$

yields the approximation

$$\hat{p} = \frac{\mathbb{E}\left[u'(W+q) + \frac{1}{2} \cdot u'''(W+q) \cdot \mathbb{V}(S \mid W)\right] \cdot [\mathbb{E}(W) + q]}{\mathbb{E}\left\{\left[u'(W+q) + \frac{1}{2} \cdot u'''(W+q) \cdot \mathbb{V}(S \mid W)\right] \cdot (W+q)\right\}} - 1 \tag{14}$$

to the equity premium.

The problem would be a trivial extension of the previous results, were it not for the dependence of q on the distribution of S via the equality

$$q = \mathbb{E}[u'(W+q+S) \cdot (W+q)].$$

The purpose of this paper is not to develop the general comparative statics of this dependence, but to determine how that dependence affects the effect of the distribution of S on the equity premium.

[17] This is why q, the price of (risky) equity, still appears in the following equation.

5.3 Idiosyncratic risk and the equity premium

Maintaining the parameterization $\mathbb{V}(S \mid W = w) = \sigma^2 w^\eta$, and in order to establish unambiguous language, let us denote the right-hand side of Eq. (14) as the function

$$\Pi(q,\sigma) = \frac{\mathbb{E}\left[u'(W+q) + \frac{\sigma^2}{2}\cdot u'''(W+q)\cdot W^\eta\right]\cdot[\mathbb{E}(W)+q]}{\mathbb{E}\left\{\left[u'(W+q) + \frac{\sigma^2}{2}\cdot u'''(W+q)\cdot \frac{\sigma^2}{2}\right]\cdot(W+q)\right\}} - 1.$$

In what follows, we will say that *the equity premium is decreasing in the price of the asset* if $\partial\Pi/\partial q < 0$, and that it is *partially decreasing in* σ if $\partial\Pi/\partial\sigma < 0$. If, on the other hand, we say that *the premium is decreasing* σ, we will refer to the overall effect

$$\frac{\mathrm{d}\hat{p}}{\mathrm{d}\sigma} = \frac{\partial\Pi}{\partial q}\cdot q' + \frac{\partial\Pi}{\partial\sigma}, \tag{15}$$

where q' results from implicitly differentiating

$$q = \mathbb{E}[u'(W+q+S)\cdot(W+q)]$$

with respect to σ.

THEOREM 10. *Suppose that the price is increasing in* σ.

1. *If the premium is decreasing in the asset price, then a necessary condition for the premium to be non-decreasing in* σ *is that*
$$\frac{\mathbb{C}\mathrm{ov}(u'(W+q),W)}{\mathbb{E}(u'(W+q))} > \frac{\mathbb{C}\mathrm{ov}(u'''(W+q)\cdot W^\eta,W)}{\mathbb{E}(u'''(W+q)\cdot W^\eta)}. \tag{16}$$

2. *If, on the other hand, the premium is non-decreasing in* q*, then Eq.* (16) *suffices for* $\hat{p}$ *to be increasing in* σ*.*

Since some of the expressions that follow are lengthy, we will sometimes write the random variable $u^{[n]}(W+q)$ simply as $U^{[n]}$. For example, Eq (16) becomes

$$\frac{\mathbb{C}\mathrm{ov}(U',W)}{\mathbb{E}(U')} > \frac{\mathbb{C}\mathrm{ov}(U'''\cdot W^\eta,W)}{\mathbb{E}(U'''\cdot W^\eta)}.$$

Note that the tension between the direct effect of q and the direct effect of σ arises when the former is negative. A complication when trying to determine the sign of the latter effect is that it involves the response of the third derivative of the Bernoulli function. Instead of attempting a full characterization, we find sufficient conditions:

THEOREM 11. *Suppose that the Bernoulli function is risk-averse of order 4. The premium is decreasing in asset price* q *if*

$$\mathbb{C}\mathrm{ov}(U'''\cdot W^\eta, W) \le 0 \tag{17}$$

and

$$\min\left\{\frac{\mathbb{C}\mathrm{ov}(U',W)}{\mathbb{E}(U')}, \frac{\mathbb{C}\mathrm{ov}(U'''\cdot W^\eta,W)}{\mathbb{E}(U'''\cdot W^\eta)}\right\} \ge \max\left\{\frac{\mathbb{C}\mathrm{ov}(U'',W)}{\mathbb{E}(U'')}, \frac{\mathbb{C}\mathrm{ov}(U^{[4]}\cdot W^\eta,W)}{\mathbb{E}(U^{[4]}\cdot W^\eta)}\right\}. \tag{18}$$

We these two previous results at hand, we can attempt to determine how the fact that the equity is long-lived affects our results in Section. 2. For the sake of simplicity, we concentrate on the cases where the volatility of idiosyncratic risk did not have an effect on the equity premium.

5.4 Homoskedastic risk and CARA preferences

Consider first the case of homoskedastic idiosyncratic risk, with $\eta = 0$ and $\mathbb{V}(S \mid W) = \sigma^2$ almost surely on W, and suppose that the Bernoulli function is exponential.

We already observed that this preferences are risk-averse of degree four. We can also pin down the sign of the effect of σ on q, as follows:

LEMMA 1. *Suppose that u displays CARA. Then, the price of the asset is increasing in σ.*

Theorems 10 and 11 immediately imply the following two results for this case:

1. the premium is decreasing in asset price q if

$$\min\left\{\frac{\mathbb{C}\mathrm{ov}(U',W)}{\mathbb{E}(U')},\frac{\mathbb{C}\mathrm{ov}(U''',W)}{\mathbb{E}(U''')}\right\} \geq \max\left\{\frac{\mathbb{C}\mathrm{ov}(U'',W)}{\mathbb{E}(U'')},\frac{\mathbb{C}\mathrm{ov}(U^{[4]},W)}{\mathbb{E}(U^{[4]})}\right\}.$$

2. if the premium is decreasing in asset price, then a necessary condition for it to be non-decreasing in σ is that

$$\frac{\mathbb{C}\mathrm{ov}(U',W)}{\mathbb{E}(U')} > \frac{\mathbb{C}\mathrm{ov}(U''',W)}{\mathbb{E}(U''')}.$$

(To be sure, note that the condition that $\mathbb{C}\mathrm{ov}[U''',W] \leq 0$, which specializes Eq. (17) to the case at hand, does not need to be assumed explicitly, as it is implied by the fact that $u^{[4]} < 0$.)

These two insights imply the following result. The idea is that under CARA preferences, the first of the two statements above holds true. This implies, as per the second statement, a necessary condition that is known to be violated.

THEOREM 12. *Suppose that the Bernoulli function displays CARA and the absolute idiosyncratic shock is homoskedastic. Then, the equity premium $\hat{p}$ is decreasing in σ.*

5.5 Homoskedastic relative risk and CRRA preferences

Considering now the case where $\mathbb{V}(S/w \mid W = w) = \sigma$, namely that $\eta = 2$, and suppose that the derivative of the Bernoulli function is homogeneous.

As before, we already know that these preferences are risk-averse of degree 4, but before we apply our general results to this case we need to determine the effects of σ on the price of equity:

LEMMA 2. *Suppose that the Bernoulli function displays CRRA. Then, the price of the asset is increasing in σ.*

Unfortunately, the assumption that $\eta = 2$, which is needed to make the relative shock S/W homoskedastic, introduces ambiguity in the sign of terms of the form

$$\mathbb{C}\mathrm{ov}[u^{[n]}(W+q)\cdot\mathbb{V}(S \mid W), W].$$

For $n = 3$, for example, the term $u^{[n]}(w+q)$ is decreasing in w, but the term $\mathbb{V}(S \mid W = w)$ is increasing, while for $n = 4$ both terms are increasing but the first one is negative. With diffidence, we resolve these ambiguities by assuming that

$$q \leq \frac{\min\{\rho,1\}}{2}\cdot\inf\mathcal{W}. \tag{19}$$

This is, indeed, an assumption on an endogenous variable, but this seems unavoidable since q appears in the arguments of both functions. Once the condition is imposed, the following is true:

Lemma 3. *Suppose that the Bernoulli function displays CRRA and Eq.* (19) *holds true. Then, the premium is decreasing in asset price q .*

These two insights imply the following result:

Theorem 13. *Suppose that the Bernoulli function displays CRRA, the relative idiosyncratic shock is homoskedastic, and Eq.* (19) *holds true. Then, the equity premium $\hat{p}$ is decreasing in σ.*

The strategy for the proof is the same as in Theorem 12: under the assumption of the theorem, Lemmas 2 and 3 give us that all the conditions that make Eq. (16) necessary for $\hat{p}$ to be non-decreasing in σ are satisfied. We argue that, nonetheless, Eq. (16) itself fails.

6 Concluding Remarks

In this paper, we have tried to further our understanding of the effects of the presence of uninsurable idiosyncratic risk on the relative price of the equity of an economy. The motivation comes from both the early work by [15] and [23], but also from the observation that various works have obtained different results because of differences in their settings.

Our first set of results focuses on the effects of the volatility of the idiosyncratic risk. After observing that the agents' prudence creates a mechanism for these effects, as [15] had pointed out, we emphasize that the cyclicality of that volatility and the behavior of the agents' risk aversion make a difference. If the agents display constant absolute risk aversion and the absolute idiosyncratic shock is homoskedastic, the volatility of this shock does not affect the equity premium; under these preferences, a countercyclical variance of the idiosyncratic shock is necessary and sufficient for the equity premium to be higher due to the presence of this shock. If it is the ratio of the idiosyncratic and the aggregate shocks that is homoskedastic, then a similar conclusion applies, *mutatis mutandis*, when the preferences display constant relative risk aversion. The literature had observed this latter result, in [23] and [12], so we see our results as complementary to those.

The second set of results relates to the effect of higher moments of the distribution of the idiosyncratic shocks. Again, whether these moments affect the equity premium and how depends on the behavior of the agents' coefficients of risk aversion and the type of cyclicality followed by the volatility of the idiosincratic shock. For example, consider this time the case of preferences displaying constant relative risk aversion. If the relative shock is homoskedastic, once again the higher moments of the distribution of idiosyncratic risk are immaterial for the level of the equity premium, whereas, if it is the absolute shock that is homoskedastic, a negatively skewed, leptokurtic distribution generates higher equity premium. These effects are driven by the fourth and fifth derivatives of the agents' utility function, which [23] had already noted. In that sense, our contribution is the identification of which features of the distribution of idiosyncratic shocks interact with which derivatives to impact the equity premium.

The previous results are obtained for a two-period economy in which, by design, the equity of the economy is short-lived and pays dividend only. In order to consider long-lived equity, where the future resale value of an asset naturally affects the present price, we adapt our analysis to a two-generation OLG economy, which allows us to compare our results with those in [22]. Here, the mathematics is more convoluted and further assumptions are necessary. Both for CARA and CRRA preferences, we consider the case in which if the asset was short-lived, the variance of the idiosyncratic shock would have no effect in the equity premium. Our results are that in those same cases, the variance has a negative effect on the premium if the asset is long-lived. We conjecture that, in general, if the equity is long-lived, the derivative of the premium with respect to the variance is lower than when the asset is short-lived.

In all these results, all the agents have been ex-ante heterogeneous and we have studied only the effect of ex-post heterogeneity.[18] Ex-ante homogeneity is useful in that it simplifies the equilibrium trade in assets, and hence their equilibrium price. In a companion paper we make progress on the study of the same problem under ex-ante heterogeneity. Still, our results allow us to say something in a more general setting.

Consider an infinite-horizon Markovian economy. Let W and S represent the next period aggregate and idiosyncratic shocks, while $\tilde{W}$ and $\tilde{S}$ denote the present variables.

Given $\tilde{W} = \tilde{w}$ and $\tilde{S} = \tilde{s}$, suppose that each the agent maximizes over y

$$v(\bar{w} - q \cdot y) + \beta v(u^{-1}(\mathbb{E}[u(W + S + W \cdot y) \mid \tilde{W} = \tilde{w}])),$$

where $\bar{w} = \tilde{w}(1 + y_-) + \tilde{s}$.[19] Since this economy is recursive, suppose that the maximum of this problem defines $u(\bar{w})$, which is the *continuation value* of starting a period with wealth $\bar{w}$.

If we assume that function v is linear, we can maintain the feature that there is no asset trade at equilibrium, which means that the equity premium in the present is the random variable

$$\tilde{P} = \frac{\mathbb{E}[u'(W + S) \mid \tilde{W}] \cdot \mathbb{E}(W \mid \tilde{W})}{\mathbb{E}[u'(W + S) \cdot W \mid \tilde{W}]} - 1.$$

In this setting our results generalize to almost sure effects on $\tilde{P}$. This observation is useful, if the assumption on function v is tenable, as it extends our analysis from Selden preferences to the Epstein-Zin framework of [7].

Appendix: Proofs

Proof of Theorem 4: The computation of the two limits and the equivalence of (a) and (b) are straightforward, so we only need to prove that (b) and (c) are equivalent.

As in the proof of Theorem 1, we can rewrite

$$\hat{p}_3 = \frac{N_2 + \frac{1}{6} \cdot \mathbb{E}[u^{[4]}(W) \cdot W^{3\eta/2}] \cdot \mathbb{E}(W) \cdot \sigma^3 \gamma}{D_2 + \frac{1}{6} \cdot \mathbb{E}[u^{[4]}(W) \cdot W^{3\eta/2} \cdot W] \cdot \sigma^3 \gamma} - 1,$$

where N_2 and D_2 are, respectively, the numerator and the denominator in the definition of $\hat{p}_2$, as in Eq. (11). By direct computation, $\partial \hat{p}_3 / \partial \gamma \geq 0$ if, and only if,

$$\mathbb{E}[u^{[4]}(W) \cdot W^{3\eta/2}] \cdot \mathbb{E}(W) \cdot D_2 \geq \mathbb{E}[u^{[4]}(W) \cdot W^{3\eta/2+1}] \cdot N_2. \tag{20}$$

With $u^{[4]} < 0$, by assumption, and since $D_2 > 0$, it follows that

$$\mathbb{E}[u^{[4]}(W) \cdot W^{3\eta/2} \cdot W] \cdot D_2 < 0.$$

Using this inequality, Eq. (20) is equivalent to

$$\frac{\mathbb{E}[u^{[4]}(W) \cdot W^{3\eta/2}] \cdot E(W)}{\mathbb{E}[u^{[4]}(W) \cdot W^{3\eta/2} \cdot W]} \leq \frac{N_2}{D_2}. \qquad \square$$

[18] In the OLG economy, all the *young* agents are homogeneous.

[19] In this equation, y_- denotes the amount of asset that the carries from last period.

Proof of Theorem 5: Once again, the two limits are straightforward. By direct computation, $\partial \hat{p}_3/\partial\sigma^2 \geq 0$ if, and only if, the sum of the following three terms is non-positive:

$$\mathbb{E}[u'(W)\cdot W]\cdot\mathbb{E}[u'''(W)\cdot W^{\eta}] - \mathbb{E}[u'(W)]\cdot\mathbb{E}[u'''(W)\cdot W^{\eta+1}]$$
$$\tfrac{\sigma\gamma}{2}\cdot\left\{\mathbb{E}[u'(W)\cdot W]\cdot\mathbb{E}[u^{[4]}(W)\cdot W^{3\eta/2}] - \mathbb{E}[u'(W)]\cdot\mathbb{E}[u^{[4]}(W)\cdot W^{3\eta/2+1}]\right\}$$
$$\tfrac{\sigma^{3/2}\gamma}{12}\cdot\left\{\mathbb{E}[u'''(W)\cdot W^{\eta+1}]\cdot\mathbb{E}[u^{[4]}(W)\cdot W^{3\eta/2}] - \mathbb{E}[u'''(W)\cdot W^{\eta}]\cdot\mathbb{E}[u^{[4]}(W)\cdot W^{3\eta/2+1}]\right\}.$$

The first of these terms is non-positive if, and only if,

$$\frac{\mathbb{E}[u'''(W)\cdot W^{\eta}]}{\mathbb{E}[u'''(W)\cdot W^{\eta+1}]} \geq \frac{\mathbb{E}[u'(W)]}{\mathbb{E}[u'(W)\cdot W]}.$$

Since $\sigma\gamma \leq 0$, the second term is non-positive if, and only if,

$$\frac{\mathbb{E}[u^{[4]}(W)\cdot W^{3\eta/2}]}{\mathbb{E}[u^{[4]}(W)\cdot W^{3\eta/2+1}]} \geq \frac{\mathbb{E}[u'(W)]}{\mathbb{E}[u'(W)\cdot W]}.$$

And since $\sigma^{3/2}\gamma \leq 0$, the third term is non-positive if, and only if,

$$\frac{\mathbb{E}[u^{[4]}(W)\cdot W^{3\eta/2}]}{\mathbb{E}[u^{[4]}(W)\cdot W^{3\eta/2+1}]} \geq \frac{\mathbb{E}[u'''(W)\cdot W^{\eta}]}{\mathbb{E}[u'''(W)\cdot W^{\eta+1}]}$$

The two inequalities in Eq. (8) suffice for these last three conditions, and hence for the sum of the three terms to be non-positive. □

Proof of Theorem 6: The argument is largely computational. Suppose one has the function

$$\phi(x,y) = \frac{a + bx + cx^{3/2}y}{d + ex + fx^{3/2}y},$$

with $a, b, d, e \geq 0$, $c, f \leq 0$, $x > 0$ and $y \leq 0$. By direct computation,

$$\frac{\partial^2\phi}{\partial y\partial x}(x,y) \geq 0$$

if, and only if,

$$\tfrac{3}{2}(cd - af)(d + ex - fx^{3/2}y)x^{1/2} + \tfrac{1}{2}(ce - bf)(e - fx^2y)x^{5/2} \geq 0. \tag{21}$$

The two inequalities in Eq. (8) guarantee that $cd \leq af$ and $ce \leq bf$. In order to guarantee Eq. (21), it then suffices that $d + ex \leq fx^{3/2}y$ and $e \leq fx^2y$. Recalling that $f \leq 0$, Eq. (9) delivers these two inequalities.

When both inequalities in Eq. (8) hold, Eq. (21) fails if $d + ex \geq fx^{3/2}y$ and $e \geq fx^2y$, which amount to Eq. (10).

The rest of the argument is similar. □

Proof of Theorem 7: For this type of preferences,

$$\hat{p}_3 = \frac{\mathbb{E}[e^{-\alpha W}\cdot(1 + \frac{\sigma^2}{2}\alpha^2\, W^{\eta} - \frac{\sigma^3}{6}\alpha^3\gamma W^{3\eta/2})]\cdot\mathbb{E}(W)}{\mathbb{E}\left[e^{-\alpha W}\cdot(1 + \frac{\sigma^2}{2}\alpha^2 W^{\eta} - \frac{\sigma^3}{6}\alpha^3\gamma W^{3\eta/2})\cdot W\right]} - 1.$$

The first claim is straightforward: if $\eta = 0$, after canceling constants,

$$\hat{p}_3 = \frac{\mathbb{E}(e^{-\alpha W})\cdot\mathbb{E}(W)}{\mathbb{E}\left(e^{-\alpha W}\cdot W\right)} - 1 = \hat{p}_2 = \bar{p}.$$

When $\eta = 2$, for the second claim, the argument resembles the proof of Theorem 2.

By Theorem 9, in order to show that $\hat{p}_3$ is increasing in γ, it suffices to argue that

$$\frac{\mathbb{E}[u^{[4]}(W) \cdot W^3] \cdot \mathbb{E}(W)}{\mathbb{E}[u^{[4]}(W) \cdot W^4]} - 1 < \hat{p}_2.$$

By direct computation, this is equivalent to

$$\mathbb{E}(e^{-\alpha W} \cdot W^3) \cdot \mathbb{E}\left[(1 + \tfrac{\sigma^2}{2}\alpha^2 W^2) \cdot e^{-\alpha W} \cdot W\right] - \mathbb{E}(e^{-\alpha W} \cdot W^4) \cdot \mathbb{E}\left[(1 + \tfrac{\sigma^2}{2}\alpha^2 W^2) \cdot e^{-\alpha W}\right] < 0.$$

Letting random variable V be i.i.d. with W, the left-hand side of this expression rewrites as

$$\mathbb{E}\left[e^{-\alpha(V+W)} \cdot (1 + \tfrac{\sigma^2}{2}\alpha^2 V) \cdot (V - W) \cdot W^3\right].$$

This number is proportional, by a factor of $1/2\,\Pr(V \neq W) > 0$, to

$$\mathbb{E}\left\{e^{-\alpha(V+W)} \cdot \left[(W^3 - V^3) + \tfrac{\sigma^2}{2}\alpha^2 V^2 W^2 (W - V)\right] \cdot (V - W) \mid V > W\right\} < 0.$$

To re-determine the effect of σ, by Theorem 5, it suffices to show that

$$\frac{\mathbb{E}(e^{-\alpha W})}{\mathbb{E}(e^{-\alpha W} \cdot W)} \geq \frac{\mathbb{E}(e^{-\alpha W} \cdot W^2)}{\mathbb{E}(e^{-\alpha W} \cdot W^3)} \geq \frac{\mathbb{E}(e^{-\alpha W} \cdot W^3)}{\mathbb{E}(e^{-\alpha W} \cdot W^4)}.$$

For the first inequality, the same technique used in the previous theorems, with random variable V being i.i.d. with W, allows us to show that

$$\mathbb{E}(e^{-\alpha W} \cdot W^2) \cdot \mathbb{E}(e^{-\alpha W} \cdot W) - \mathbb{E}(e^{-\alpha W}) \cdot \mathbb{E}(e^{-\alpha W} \cdot W^3)$$

is proportional, by a positive factor, to

$$\mathbb{E}\left[e^{-\alpha(W+V)} \cdot (W^2 - V^2) \cdot (V - W) \mid W > V\right] \leq 0.$$

For the second inequality,

$$\mathbb{E}(e^{-\alpha W} \cdot W^3)^2 - \mathbb{E}(e^{-\alpha W} \cdot W^4) \cdot \mathbb{E}(e^{-\alpha W} \cdot W^2)$$

is directly proportional to

$$\mathbb{E}\left[e^{-\alpha(W+V)} \cdot W^2 \cdot V^2 \cdot (-W^2 + 2VW - V^2) \mid W > V\right] \leq 0.$$

This result on the cross derivative follows immediately from Theorem 6, given that both inequalities in Eq. (8) fail, as seen above. $\square$

Proof of Theorem 8: For the first claim, note that the proof of Theorem 3 implies that

$$\frac{\mathbb{E}(W^{-\rho+1})}{\mathbb{E}(W^{-\rho})} > \frac{\mathbb{E}(W^{-\rho-1})}{\mathbb{E}(W^{-\rho-2})}.$$

By the same argument,

$$\frac{\mathbb{E}(W^{-\rho-1})}{\mathbb{E}(W^{-\rho-2})} > \frac{\mathbb{E}(W^{-\rho-2})}{\mathbb{E}(W^{-\rho-3})}.$$

It follows that

$$\frac{\mathbb{E}[u'(W) \cdot W]}{\mathbb{E}[u'(W)]} > \frac{\mathbb{E}[u'''(W) \cdot W]}{\mathbb{E}[u'''(W)]} > \frac{\mathbb{E}[u^{[4]}(W) \cdot W]}{\mathbb{E}[u^{[4]}(W)]},$$

and hence that

$$\mathbb{E}[u'(W)\cdot W]\cdot\mathbb{E}[u^{[4]}(W)] > \mathbb{E}[u'(W)]\cdot\mathbb{E}[u^{[4]}(W)\cdot W]$$

and

$$\mathbb{E}[u'''(W)\cdot W]\cdot\mathbb{E}[u^{[4]}(W)] > \mathbb{E}[u'''(W)]\cdot\mathbb{E}[u^{[4]}(W)\cdot W].$$

Aggregating,

$$\left\{\mathbb{E}\left[u'(W)+\tfrac{\sigma^2}{2}\cdot u'''(W)\right]W\right\}\cdot\mathbb{E}\left[u^{[4]}(W)\right] > \mathbb{E}\left[u'(W)+\tfrac{\sigma^2}{2}\cdot u'''(W)\right]\cdot\mathbb{E}\left[u^{[4]}(W)\cdot W\right],$$

which implies, by Theorem 4, that $\hat{p}_3$ is decreasing in γ.

To see that $\hat{p}_3$ is increasing in σ, we prove that condition (8) holds and invoke Theorem 5. To see that

$$\frac{\mathbb{E}[u'''(W)\cdot W^\eta]}{\mathbb{E}\left[u'''(W)\cdot W^{\eta+1}\right]} \geq \frac{\mathbb{E}[u'(W)]}{\mathbb{E}[u'(W)\cdot W]},$$

we just need to argue that

$$\mathbb{E}(W^{\eta-\rho-2})\cdot\mathbb{E}(W^{1-\rho}) - \mathbb{E}(W^{\eta-\rho-1})\cdot\mathbb{E}(W^{-\rho}) \geq 0.$$

Using, as before, an ancillary random variable V that is i.i.d. with W, the latter expectation is directly proportional to

$$\mathbb{E}\left[W^{-\rho}\cdot V^{-\rho}\cdot(V-W)\cdot(W^{-1}-V{-1})\cdot(W^{-1}+V{-1})\mid W>V\right] \geq 0.$$

The sign of the cross derivative follow from Theorem (6), upon substitution, since we just proved that condition (8) holds true in this case.

As for the second claim, by direct computation,

$$\hat{p}_3 = \frac{\mathbb{E}(W^{-\rho})\cdot\mathbb{E}(W)\cdot[1+\rho(\rho+1)\frac{\sigma^2}{2}+\rho(\rho+1)(\rho+2)\frac{\sigma^{3/2}}{6}\gamma]}{\mathbb{E}(W^{-\rho+1})\cdot[1+\rho(\rho+1)\frac{\sigma^2}{2}+\rho(\rho+1)(\rho+2)\frac{\sigma^{3/2}}{6}\gamma]} - 1 = \hat{p}_2. \qquad \square$$

Proof of Theorem 9: This proof is a direct generalization of previous arguments, so we omit it. $\square$

Proof of Theorem 10: Recall Eq. (15). For the first result, note that the first summand on the right-hand side of the last expression is negative, so a necessary condition for the sum to be positive is that the second summand be positive. For the second result, under the assumptions the first summand is non-negative, so the sum is positive if so is the second summand.

In both cases, all one needs is that $\partial\Pi/\partial\sigma > 0$. The proof that this inequality is equivalent to Eq. (16) closely resembles to the argument for Theorem 1, so we omit it. $\square$

Proof of Theorem 11: We can write Eq. (14) as

$$\hat{p} = \frac{f(q)+\sigma^2\cdot g(q)}{\varphi(q)+\sigma^2\cdot\gamma(q)},$$

where

$$f(q) = \mathbb{E}[u'(W+q)]\cdot[E(W)+q],$$
$$g(q) = \tfrac{1}{2}\mathbb{E}[u'''(W+q)\cdot W^\eta]\cdot[E(W)+q],$$
$$\varphi(q) = \mathbb{E}[u'(W+q)\cdot(W+q)]$$

and

$$\gamma(q) = \tfrac{1}{2}\mathbb{E}[u'''(W+q)\cdot W^\eta\cdot(W+q)].$$

With this formulation, $\hat{p}$ is decreasing in q if, and only if,

$$[f'(q) + \sigma^2 \cdot g'(q)] \cdot [\varphi(q) + \sigma^2 \cdot \gamma(q)] < [\varphi'(q) + \sigma^2 \cdot \gamma'(q)] \cdot [f(q) + \sigma^2 \cdot g(q)],$$

which holds true if

$$f'(q) \cdot \varphi(q) < \varphi'(q) \cdot f(q) \tag{22}$$
$$f'(q) \cdot \gamma(q) \leq \varphi'(q) \cdot g(q) \tag{23}$$
$$g'(q) \cdot \varphi(q) \leq \gamma'(q) \cdot f(q) \tag{24}$$
$$g'(q) \cdot \gamma(q) \leq \gamma'(q) \cdot g(q). \tag{25}$$

Upon substitution, Eq. (22) is equivalent to

$$\big\{\mathbb{E}(U'')[\mathbb{E}(W) + q] + \mathbb{E}(U')\big\} \cdot \mathbb{E}[U' \cdot (W + q)] < \big\{\mathbb{E}[U'' \cdot (W + q)] + \mathbb{E}(U')\big\} \cdot E(U') \cdot [\mathbb{E}(W) + q],$$

which is, by direct computation,

$$\big\{\mathbb{E}(U'') \cdot \mathbb{E}[U' \cdot (W + q)] - \mathbb{E}(U') \cdot \mathbb{E}[U'' \cdot (W + q)]\big\} \cdot [\mathbb{E}(W) + q] + \mathbb{E}(U') \cdot \mathbb{C}\mathrm{ov}(U', W) < 0. \tag{$*$}$$

Since $u' > 0$ and $u'' < 0$, we have that $\mathbb{E}(U') > 0$ and $\mathbb{C}\mathrm{ov}(U', W) < 0$, so it suffices that

$$\mathbb{E}(U'') \cdot \mathbb{E}[U' \cdot (W + q)] \leq \mathbb{E}(U') \cdot \mathbb{E}[U'' \cdot (W + q)],$$

for inequality $(*)$ to hold, as $\mathbb{E}(W) + q > 0$. As in the proof of Theorem 1, this is equivalent to

$$\frac{\mathbb{C}\mathrm{ov}(U', W)}{\mathbb{E}(U')} \geq \frac{\mathbb{C}\mathrm{ov}(U'', W)}{\mathbb{E}(U'')},$$

which is one of the inequalities that are part of Eq. (18).

Similarly, Eq. (23) is equivalent to the requirement that the sum of

$$\big\{\mathbb{E}(U'') \cdot \mathbb{E}[U''' \cdot W^\eta \cdot (W + q)] - \mathbb{E}[U''' \cdot W^\eta] \cdot \mathbb{E}[U'' \cdot (W + q)]\big\} \cdot [\mathbb{E}(W) + q] \tag{$**$}$$

and

$$\mathbb{E}(U') \cdot \mathbb{C}\mathrm{ov}[U''' \cdot W^\eta, W] \tag{$***$}$$

be non-positive.

Since $u' > 0$ and $\mathbb{C}\mathrm{ov}[U''' \cdot W^\eta, W] \leq 0$, we have that the expression in $(***)$ is non-positive. On the other hand, since $\mathbb{E}(W) + q > 0$, for inequality $(**)$ to hold it suffices that

$$\mathbb{E}(U'') \cdot \mathbb{E}[U''' \cdot W^\eta \cdot (W + q)] \leq \mathbb{E}[U''' \cdot W^\eta] \cdot \mathbb{E}[U'' \cdot (W + q)],$$

which is equivalent to

$$\frac{\mathbb{C}\mathrm{ov}[U''' \cdot W^\eta, W]}{\mathbb{E}[U''' \cdot W^\eta]} \geq \frac{\mathbb{C}\mathrm{ov}(U'', W)}{\mathbb{E}(U'')}.$$

For Eqs. (22) and (23) to hold true, it thus suffices that

$$\min\left\{\frac{\mathbb{C}\mathrm{ov}(U', W)}{\mathbb{E}(U')}, \frac{\mathbb{C}\mathrm{ov}[U''' \cdot W^\eta, W]}{\mathbb{E}[U''' \cdot W^\eta]}\right\} \geq \frac{\mathbb{C}\mathrm{ov}(U'', W)}{\mathbb{E}(U'')}.$$

By a virtually identical analysis, using that $u''' > 0$ and $u^{[4]} < 0$, one can prove that

$$\min\left\{\frac{\mathbb{C}\mathrm{ov}(U', W)}{\mathbb{E}(U')}, \frac{\mathbb{C}\mathrm{ov}[U''' \cdot W^\eta, W]}{\mathbb{E}[U''' \cdot W^\eta]}\right\} \geq \frac{\mathbb{C}\mathrm{ov}[U^{[4]} \cdot W^\eta, W]}{\mathbb{E}[U^{[4]} \cdot W^\eta]}$$

suffices for Eqs. (24) and (25) $\square$

Proof of Lemma 1: Let $u(w) = -e^{-\alpha w}$, for some $\alpha > 0$. Then, $u^{[n]}(w) = (-\alpha)^n u(w)$ and

$$q = \mathbb{E}\left[u'(W+q+S)\cdot(W+q)\right] = e^{-\alpha q}\cdot\left\{\mathbb{E}[u'(W+S)\cdot W] + \mathbb{E}[u'(W+S)]\cdot q\right\},$$

so

$$e^{\alpha q} = \frac{\mathbb{E}[u'(W+S)\cdot W]}{q} + \mathbb{E}[u'(W+S)].$$

This expression is transcendental, so we can *only* obtain q' by implicit differentiation:

$$\left\{e^{\alpha q} + \frac{\mathbb{E}[u'(W+S)\cdot W]}{q^2}\right\}\cdot q' = \frac{\partial}{\partial\sigma}\left\{\frac{\mathbb{E}[u'(W+S)\cdot W]}{q} + \mathbb{E}[u'(W+S)]\right\}.$$

Since exponential preferences are strictly increasing and strictly prudent, we know that

$$\mathbb{E}[u'(W+S)\cdot W] = \mathbb{E}\left\{\mathbb{E}[u'(W+S)\mid W]\cdot W\right\}$$

and

$$\mathbb{E}[u'(W+S)] = \mathbb{E}\left\{\mathbb{E}[u'(W+S)\mid W]\right\}$$

are both increasing in σ, which implies that $q' > 0$. □

Proof of Theorem 12: Let $u(w) = -e^{-\alpha w}$, for some $\alpha > 0$. Again, $u^{[n]}(w) = (-\alpha)^n u(w)$, which implies that

$$\mathbb{C}\mathrm{ov}[u^{[n]}(W+q), W] = (-\alpha)^n \mathbb{C}\mathrm{ov}[u(W+q), W]$$

and

$$\mathbb{E}[u^{[n]}(W+q)] = (-\alpha)^n \mathbb{E}[u(W+q)].$$

It follows that

$$\frac{\mathbb{C}\mathrm{ov}[u^{[n]}(W+q), W]}{\mathbb{E}[u^{[n]}(W+q)]} = \frac{\mathbb{C}\mathrm{ov}[u(W+q), W]}{\mathbb{E}[u(W+q)]}$$

for all orders of differentiation. Theorem 11 implies that premium $\hat{p}$ is decreasing in q, while Lemma 1 tells us that the price of the asset is increasing in σ. It then follows from Theorem 10 that condition (16), which does not hold true, is necessary for $\hat{p}$ to be non-decreasing in Σ. □

Proof of Lemma 2: We obtain this result, once again, by implicitly differentiating

$$q = u'(1)\cdot\mathbb{E}[(W+q+S)^{-\rho}\cdot(W+q)]$$

with respect to σ. By the implicit function theorem q' equals the product of

$$\frac{u'(1)}{\mathbb{E}\left[1+\rho(W+q+S)^{-(\rho+1)}\cdot(W+q) - (W+q+S)^{-\rho}\right]} \tag{$*$}$$

and

$$\mathbb{E}\left\{\frac{\partial}{\partial\sigma}\mathbb{E}[(W+q+S)^{-\rho}\mid W]\cdot(W+q)\right\}, \tag{$**$}$$

so long as the denominator on the former expression is non-zero. We actually want that denominator to be strictly positive, which is the case, since $\rho > 0$, $W+q > 0$ with probability one by assumption, and

$$\rho(w+q) > 0 \Leftrightarrow 1 + \frac{\rho(w+q)}{(w+q+s)^{\rho+1}} > \frac{1}{(w+q+s)^{\rho}}.$$

Since $u'(1) > 0$, it follows that the term in Eq. $(*)$ is strictly positive.

That the term in Eq. $(**)$ is also positive is immediate, since $(w+q+s)^{-\rho}$ is strictly convex in s, and an increase in σ is a mean-preserving spread of S given W. □

Proof of Lemma 3: By Theorem 11, we just need to argue that Eqs. (17) and (18) are satisfies under the premises of the lemma. For simplicity, we divide the proof in a series of claims:

Claim 1. *Eq.* (17) *is satisfied*

Proof. It suffices for us to argue that $u'''(W+q) \cdot \mathbb{V}(S \mid W)$ and W are anti-comonotone with probability one. Letting the function

$$w \mapsto u'''(w+q) \cdot \mathbb{V}(S \mid W = w) = \sigma\rho(\rho+1)u'(1)w^2,$$

we have that this mapping is non-increasing so long as $w \geq 2q/\rho$. Since $q \leq {}^{\rho}\!/_{2} \inf \mathcal{W}$, by assumption, we have that this inequality holds with probability 1. $\square$

Claim 2. *The following two inequalities, which are part of Eq.* (18)*, also hold:*

$$\frac{\mathbb{C}\mathrm{ov}(U', W)}{\mathbb{E}(U')} \geq \frac{\mathbb{C}\mathrm{ov}(U'', W)}{\mathbb{E}(U'')}$$

and

$$\frac{\mathbb{C}\mathrm{ov}[U''' \cdot \mathbb{V}(S \mid W), W]}{\mathbb{E}[U''' \cdot \mathbb{V}(S \mid W)]} \geq \frac{\mathbb{C}\mathrm{ov}[U^{[4]} \cdot \mathbb{V}(S \mid W), W]}{\mathbb{E}[U^{[4]} \cdot \mathbb{V}(S \mid W)]},$$

Proof. We can rewrite the inequalities as

$$\frac{\mathbb{E}[(W+q)^{-\rho} \cdot W]}{\mathbb{E}[(W+q)^{-\rho}]} \geq \frac{\mathbb{E}[(W+q)^{-(\rho+1)} \cdot W]}{\mathbb{E}[(W+q)^{-(\rho+1)}]}$$

and

$$\frac{\mathbb{E}[(W+q)^{-(\rho+2)} \cdot W^3]}{\mathbb{E}[(W+q)^{-(\rho+2)} \cdot W^2]} \geq \frac{\mathbb{E}[(W+q)^{-(\rho+3)} \cdot W^3]}{\mathbb{E}[(W+q)^{-(\rho+3)} \cdot W^2]}.$$

If we define now the function

$$h(n) = \frac{\mathbb{E}[(W+q)^{-n} \cdot W^m]}{\mathbb{E}[(W+q)^{-n} \cdot W^{m-1}]}$$

over $n > 0$, given any $m \geq 0$, it suffices to show that h is non-increasing in n.

By direct computation, $h'(n) \leq 0$ if, and only if,

$$\mathbb{E}[(W+q)^{-n} \cdot W^m \cdot \ln(W+q)] \cdot \mathbb{E}[(W+q)^{-n} \cdot W^{m-1}]$$

is at least as large as

$$\mathbb{E}[(W+q)^{-n} \cdot W^{m-1} \cdot \ln(W+q)] \cdot \mathbb{E}[(W+q)^{-n} \cdot W^m].$$

Letting random variable V be i.i.d. with W, this is the requirement that

$$\mathbb{E}\left\{(W-V) \cdot (VW)^{m-1} \cdot [(V+q)(W+q)]^{-n} \cdot \ln(W+q)\right\} \geq 0.$$

This expectation is proportional, by a factor of $\Pr(V \neq W)/2$, to the sum of

$$\mathbb{E}\left\{(W-V) \cdot (VW)^{m-1} \cdot [(V+q)(W+q)]^{-n} \cdot \ln(W+q) \mid V > W\right\}$$

and

$$\mathbb{E}\left\{(W-V) \cdot (VW)^{m-1} \cdot [(V+q)(W+q)]^{-n} \cdot \ln(W+q) \mid V < W\right\}.$$

Since V and W follow the same distribution, the latter is

$$\mathbb{E}\left\{(V-W) \cdot (VW)^{m-1} \cdot [(V+q)(W+q)]^{-n} \cdot \ln(V+q) \mid V > W\right\},$$

so the sum equals

$$\mathbb{E}\left\{(W-V) \cdot (VW)^{m-1} \cdot [(V+q)(W+q)]^{-n} \cdot [\ln(W+q) - \ln(V+q)] \mid V > W\right\},$$

which is, indeed, non-negative. $\square$

Claim 3. *One more of the inequalities in Eq.* (18) *also holds:*

$$\frac{\mathbb{C}\mathrm{ov}[U''' \cdot \mathbb{V}(S \mid W), W]}{\mathbb{E}[U''' \cdot \mathbb{V}(S \mid W)]} \geq \frac{\mathbb{C}\mathrm{ov}(U'', W)}{\mathbb{E}(U'')},$$

Proof. We want to prove that

$$\frac{\mathbb{E}[(W+q)^{-(\rho+2)} \cdot W^3]}{\mathbb{E}[(W+q)^{-(\rho+2)} \cdot W^2]} \geq \frac{\mathbb{E}[(W+q)^{-(\rho+1)} \cdot W]}{\mathbb{E}[(W+q)^{-(\rho+1)}]},$$

which is equivalent to the requirement that

$$\mathbb{E}[(W+q)^{-(\rho+2)} W^3] \cdot \mathbb{E}[(W+q)^{-(\rho+1)}] \geq \mathbb{E}[(W+q)^{-(\rho+2)} W^2] \cdot \mathbb{E}[(W+q)^{-(\rho+1)} W].$$

With V defined as above, this is

$$\mathbb{E}\left\{V \cdot [(V+q)(W+q)]^{-(\rho+1)} \cdot \left(\frac{V^2}{V+q} - \frac{W^2}{W+q}\right)\right\} \geq 0,$$

or

$$\mathbb{E}\left\{(V-W) \cdot [(V+q)(W+q)]^{-(\rho+1)} \cdot \left(\frac{V^2}{V+q} - \frac{W^2}{W+q}\right) \;\middle|\; V > W\right\} \geq 0.$$

In order to guarantee this, we need to argue that

$$v > w \Rightarrow \frac{v^2}{v+q} \geq \frac{w^2}{w+q},$$

or, equivalently, that the ratio $w^2/(w+q)$ is non-decreasing for $w \in \mathcal{W}$. By direct computation, this is true since $\mathcal{W} \subseteq \mathbb{R}_{++}$ and $q \geq 0$. □

Claim 4. *The remaining inequality in Eq.* (18) *also holds:*

$$\frac{\mathbb{C}\mathrm{ov}(U', W)}{\mathbb{E}(U')} \geq \frac{\mathbb{C}\mathrm{ov}[U^{[4]} \cdot \mathbb{V}(S \mid W), W]}{\mathbb{E}[U^{[4]} \cdot \mathbb{V}(S \mid W)]}.$$

Proof. The desired inequality is

$$\frac{\mathbb{E}[(W+q)^{-\rho} \cdot W]}{\mathbb{E}[(W+q)^{-\rho}]} \geq \frac{\mathbb{E}[(W+q)^{-(\rho+3)} \cdot W^3]}{\mathbb{E}[(W+q)^{-(\rho+3)} \cdot W^2]},$$

which is equivalent to the requirement that

$$\mathbb{E}\left[(V-W) \cdot W^2 \cdot (V+q)^{-\rho} \cdot (W+q)^{-(\rho+3)}\right] \geq 0,$$

or, equivalently, that

$$\mathbb{E}\left\{(V-W) \cdot [(V+q)(W+q)]^{-\rho} \cdot \left(\frac{W^2}{(W+q)^3} - \frac{V^2}{(V+q)^3}\right) \;\middle|\; V > W\right\} \geq 0.$$

For this, it suffices that the ratio $w^2/(w+q)^3$ be non-increasing at all $w \in \mathcal{W}$. This is guaranteed, indeed, by the assumption that $q \leq 1/2 \inf \mathcal{W}$. □

The four claims together yield the hypotheses of Theorem 11. □

Proof of Theorem 13: We just need to argue that

$$\frac{\mathbb{E}[(W+q)^{-\rho} \cdot W]}{\mathbb{E}[(W+q)^{-\rho}]} \le \frac{\mathbb{E}[(W+q)^{-(\rho+2)} \cdot W^3]}{\mathbb{E}[(W+q)^{-(\rho+2)} \cdot W^2]}.$$

To see this, note again that it is equivalent to

$$\mathbb{E}[(V-W) \cdot V^2 \cdot (V+q)^{-(\rho+2)}(W+q)^{-\rho}] \ge 0,$$

or

$$\mathbb{E}\left\{(V-W) \cdot [(V+q)(W+q)]^{-\rho} \cdot \left(\frac{V^2}{(V+q)^2} - \frac{W^2}{(W+q)^2}\right) \,\middle|\, V > W\right\} \ge 0.$$

For this inequality to hold true, it suffices that $w/(w+q)$ be non-increasing at all $w \in \mathcal{W}$, which is true since $\mathcal{W} \subseteq \mathbb{R}_{++}$ and $q \ge 0$. □

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