Cooperative Ph.D. Program in the School of Economic Sciences and Finance

QUANTITATIVE METHODS QUALIFYING EXAMINATION

August 4, 2025
8:45 a.m. to 1:00 p.m.

THERE ARE SIX QUESTIONS – ANSWER ANY **TWO** QUESTIONS NUMBERED 1-3 AND ANY **TWO** QUESTIONS NUMBERED 4-6. IF YOU ANSWER ALL THREE IN EITHER GROUP, ONLY THE FIRST TWO WILL BE GRADED, THERE WILL BE A TOTAL OF FOUR QUESTIONS YOU NEED TO ANSWER, TWO OF QUESTIONS 1-3; AND TWO OF QUESTIONS 4-6.

- ✿ You **must** complete the examination within four hours. You will have 15 minutes to read over the questions before starting (8:45-9:00).

- ✿ This exam is closed book. Calculators and paper will be provided.

- ✿ Read the question carefully. Allocate your time carefully. Parts within questions will often vary in difficulty and weight. Be sure to do all parts of each question chosen.

- ✿ If necessary, it is permissible to make clarifying assumptions, but be sure to label them explicitly. (Grades will not take unstated assumptions for granted.) Also, label graphs and define notation.

- ✿ Number your answer sheets consecutively. Begin your answer to each question on a new page and identify the questions number.

- ✿ Leave ½-1" spacing around the edges of your paper.

Please write your exam ID number on the top of this page.

Statistics

Q1. Provide clear and precise definitions for each of the statistical/econometric concepts listed below. In addition, explain how they are used and their importance in statistics and econometrics research.

a) Unbiased estimators, biased estimators, variance of estimators, and how they relate to the mean square error (MSE) criterion
b) Best Linear Unbiased Estimators (BLUEs) and Minimum Variance Unbiased Estimators (MVUEs).
c) Sufficient statistics, complete sufficient statistics, and their relationship to MVUEs
d) Probability limits, consistency of an estimator, and consistency of a test statistic
e) Asymptotic Normality of an estimator
f) Type I versus Type II errors as they relate to null and alternative hypotheses
g) Size, power, and probability values of hypothesis tests, and their relevance in testing null hypotheses

Q2. A researcher has 1000 *iid* sample observations, $(x_1, x_2, ..., x_{1000})$, on the personal incomes of the residents of Upper Slobovia, where the x_i's are sampled from some Gamma population distribution,

$$f(z) = \frac{1}{\beta^\alpha \Gamma(\alpha)} z^{\alpha-1} e^{-z/\beta} I_{(0,\infty)}(z)$$

The moment generating function of the Gamma distribution is given by $(1-\beta t)^{-\alpha}$, *for* $t < \beta^{-1}$.

The researcher is considering using the sample mean, $\overline{X} = \dfrac{\sum_{i=1}^{1000} X_i}{1000}$, to estimate the mean income of the residents. *Justify* each of your answers to the following questions.

    a. What is the mean of the Gamma population distribution?

    b. Is $\overline{X}$ unbiased for estimating the population mean?

    c. What is the variance of $\overline{X}$?

    d. Is $\overline{X}$ the BLUE?

    e. Is $\overline{X}$ the MVUE?

    f. Is $\overline{X}$ a consistent estimator?

    g. What is the probability distribution of $\overline{X}$ in finite samples?

    h. Is a valid asymptotic distribution for the estimator given by

$$\overline{X} \overset{a}{\sim} N\left(\alpha\beta, \alpha\beta^2 / 1000\right)?$$

    i. The "coefficient of variation" is defined as $\dfrac{\sigma}{\mu}$, and in this case expresses the standard deviation of income as a proportion of the mean of income. The researcher is considering using the statistic $\dfrac{S}{\overline{X}}$ to estimate the coefficient of variation, where $S^2 = \sum_{i=1}^{1000}\left(X_i - \overline{X}\right)^2 / 999$. The researcher states that the estimator of the coefficient of variation is both unbiased and consistent for the true coefficient of variation, and in defense of his statement, he *claims* the following are true:

$$E\left[\frac{S}{\overline{X}}\right] = \left[\frac{E[S]}{E[\overline{X}]}\right] = \frac{\sigma}{\mu} \quad and \quad plim\left(\frac{S}{\overline{X}}\right) = \left(\frac{plim(S)}{plim(\overline{X})}\right) = \frac{\sigma}{\mu}.$$

Do you agree with either or both of the researcher's arguments about his estimator of the coefficient of variation? Explain.

Q3. Consider the general linear model $Y = x\beta + \varepsilon$ with classical assumptions: $E(Y) = x\beta$ and $E\varepsilon = [0]$, $Cov(Y) = \sigma^2 I = Cov(\varepsilon) = E\varepsilon\varepsilon'$, and x is nonrandom with rank(x)=k (i.e., full column rank).

a) Setup the least squares optimization problem for the GLM. Explicitly derive the least squares estimators $\left(\hat{\beta}, \hat{S}^2\right)$ for $\left(\beta, \sigma^2\right)$.

b) What does it mean for $\hat{\beta}$ to be an unbiased estimator? Be explicit. Demonstrate that $\hat{\beta}$ is an unbiased estimator of β.

c) Under what conditions is $\hat{\beta}$ the best linear unbiased estimator for β? Explain.

d) Derive the cov ($\hat{\beta}$). Under what conditions is $\left(\hat{\beta}, \hat{S}^2\right)$ the minimum variance unbiased estimator for $\left(\beta, \sigma^2\right)$? Explain.

e) Define the terms consistency, asymptotic normality, and asymptotic efficiency of an estimator. Under what conditions is $\hat{\beta}$ consistent, asymptotically normally distributed, and asymptotically efficient?

Econometrics

Q4. Consider the general linear model $Y = x\beta + \varepsilon$ and linear restriction $R\beta = r$. You may assume that the classical assumptions of the GLM hold and normality of model residuals.

a) Setup the objective function for the restricted maximum likelihood estimator of the GLM. State all assumptions.

b) Derive the restricted maximum likelihood estimator of the GLM. Show work.

c) How does the restricted estimator relate to the estimator $\hat{\beta} = (X'X)^{-1} X'y$? Is it more or less bias? Is it more or less efficient?

d) From parts a,b) derive a consistent and asymptotic level α Lagrangian Multiplier test of $H_0 : R\beta = r$ v $H_a : R\beta \neq r$. You should state all assumptions, derive the test statistic, its asymptotic distribution, and the appropriate critical region.

e) Suppose the residuals in the model above exhibited some form of heteroscedasticity. How would this alter your answer to part d)? Explain.

Q5. Suppose we are interested in the relationship between whether an individual owns a home, $Y \in \{0,1\}$, and whether individual i graduated from university $X \in \{0,1\}$, specifically the regression equation $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$. The researchers have a sample of 100 individuals previously entered into a university tuition waiver lottery, as well as their current university graduation and home ownership status. For clarity, Y takes a value of one if the individual owns their home, X takes a value of one if they graduated from university, and Z takes a value of one only if they won a college tuition waiver prior to university enrollment choice. Note that not every individual who wins the tuition waiver chooses to attend university. Data for the full sample of 100 individuals is below:

# Observations	Y	X	Z
20	0	0	0
10	1	0	0
10	0	1	0
10	1	1	0
5	0	0	1
5	1	0	1
15	0	1	1
25	1	1	1

Sample statistics from this data include: $\bar{Y} = 0.5$, $\bar{X} = 0.6$, $\bar{Z} = 0.5$, $\Sigma(Z_i - \bar{Z}) \cdot (Z_i - \bar{Z}) = 25$, $\Sigma(X_i - \bar{X}) \cdot (Z_i - \bar{Z}) = 10$, $\Sigma(X_i - \bar{X}) \cdot (X_i - \bar{X}) = 24$, $\Sigma(Y_i - \bar{Y}) \cdot (X_i - \bar{X}) = 5$, $\Sigma(Y_i - \bar{Y}) \cdot (Z_i - \bar{Z}) = 5$. $\bar{Y}_{z=1} = 0.6$, $\bar{Y}_{z=0} = 0.4$, $\bar{X}_{z=1} = 0.8$, $\bar{X}_{z=0} = 0.4$, $\bar{Y}_{x=1} = 0.5833$, $\bar{Y}_{x=0} = 0.375$

a) Initially, the researchers ignore the information provided by the tuition waiver lottery and instead simply estimate $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$. What must be true for the following condition to hold: $\beta_1 = E[Y|X = 1] - E[Y|X = 0]$.
Provide intuition for what this condition means in words and whether it is likely true in practice.

b) Given the model and data, provide the numerical value of the OLS estimator, $\widehat{\beta_1^{OLS}}$. <u>Provide the numerical value according to the data above up to three decimal places.</u>

c) To alleviate potential endogeneity issues, suppose now the researchers plan to incorporate data from the random tuition lottery in an instrumental variables approach. Using $Z \in \{0,1\}$ as an instrument for whether they graduated from university (X), describe what must be true for the following condition to hold:
$$\frac{E[Y|Z = 1] - E[Y|Z = 0]}{E[X|Z = 1] - E[X|Z = 0]} = \beta_1$$

Provide intuition for what this condition means in words.

d) Given the model and data, provide the numerical value of the IV estimator, $\widehat{\beta_1^{IV}}$ from part c). <u>Provide the numerical value according to the data above up to three decimal places.</u>

e) Suppose you were to conduct a test for whether the estimator in parts c) and d) β_1^{IV} is equal to the β_1^{OLS} estimator from parts a) and b). Under the null hypothesis $\beta_1^{IV} = \beta_1^{OLS}$ and the alternate hypothesis $\beta_1^{IV} \neq \beta_1^{OLS}$, you calculate a test statistic according to the following formula, $H = (\widehat{\beta_1^{OLS}} - \widehat{\beta_1^{IV}})'(\widehat{Var(\beta_1^{IV})} - \widehat{Var(\beta_1^{OLS})})^{-1}(\widehat{\beta_1^{OLS}} - \widehat{\beta_1^{IV}})$. Assuming the instrument Z (tuition waiver) is exogenous, what would a failure to reject the null hypothesis in this subpart indicate regarding the assumption in part a)? Would this raise any concerns with the assumption in part c)?

Q6: Provide clear and precise proofs for the following set of independent sub questions. Be careful to provide a formal proof and clarify every step taken and assumption used.

i) For the general regression model $Y = X\beta + u$, let β^{OLS} be the ordinary least squares estimator of β. Prove, for any alternative linear estimator $\tilde{\beta}$ of β, that:

$$\left(Y - X\tilde{\beta}\right)'\left(Y - X\tilde{\beta}\right) \geq (Y - X\beta^{OLS})'(Y - X\beta^{OLS})$$

Hint: Use the add and subtract strategy, e.g. $(Y - X\tilde{\beta}) = (Y - X\beta^{OLS} + X\beta^{OLS} - X\tilde{\beta})$

ii) In some cases, we wish to restrict the model such that the outcome variable of y takes a value of zero when the single independent variable x is zero. Suppose a researcher were to estimate a slope estimator of the form: $\tilde{y} = \widetilde{\beta_1}x$ in which $\widetilde{\beta_1}$ minimizes the sum of the squared residuals. Contrast this estimator $\widetilde{\beta_1}$ with the standard regression single variable model $y = \beta_0 + \beta_1 x + u$. Prove that $\widetilde{\beta_1}$ is an unbiased estimator of β_1 when $\beta_0 = 0$. In addition, describe other cases or conditions where $\widetilde{\beta_1}$ is an unbiased estimator of β_1.

iii) Consider the regression model in matrix notation: $Y = X_1\beta_1 + X_2\beta_2 + u$ where Y is an $n \times 1$ vector, X_1 is an $n \times k_1$ matrix, and X_2 is an $n \times k_2$ matrix. Considering the identity matrix I, prove that the $n \times n$ matrix $M_2 = I - X_2(X'_2 X_2)^{-1}X'_2$ is both symmetric ($M_2 = M_2'$) and idempotent ($M_2 M_2 = M_2$).

Cumulative Standard Normal Distribution (Table entry is $\Phi(z) = \text{Prob}[Z \le z]$)

z	0.0	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.4878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Right-Hand Tail Critical Values for the Student t-distribution

<----------------- alpha ----------------->

df	0.20	0.15	0.10	0.05	0.025	0.01	0.005	0.0025	0.001	0.0005
1	1.376	1.963	3.078	6.314	12.706	31.821	63.656	127.321	318.289	636.578
2	1.061	1.386	1.886	2.920	4.303	6.965	9.925	14.089	22.328	31.600
3	0.978	1.250	1.638	2.353	3.182	4.541	5.841	7.453	10.214	12.924
4	0.941	1.190	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	0.920	1.156	1.476	2.015	2.571	3.365	4.032	4.773	5.894	6.869
6	0.906	1.134	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	0.889	1.108	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.883	1.100	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.879	1.093	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.876	1.088	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.768
24	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.689
28	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.660
30	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
31	0.853	1.054	1.309	1.696	2.040	2.453	2.744	3.022	3.375	3.633
32	0.853	1.054	1.309	1.694	2.037	2.449	2.738	3.015	3.365	3.622
33	0.853	1.053	1.308	1.692	2.035	2.445	2.733	3.008	3.356	3.611
34	0.852	1.052	1.307	1.691	2.032	2.441	2.728	3.002	3.348	3.601
35	0.852	1.052	1.306	1.690	2.030	2.438	2.724	2.996	3.340	3.591
36	0.852	1.052	1.306	1.688	2.028	2.434	2.719	2.990	3.333	3.582
37	0.851	1.051	1.305	1.687	2.026	2.431	2.715	2.985	3.326	3.574
38	0.851	1.051	1.304	1.686	2.024	2.429	2.712	2.980	3.319	3.566
39	0.851	1.050	1.304	1.685	2.023	2.426	2.708	2.976	3.313	3.558
40	0.851	1.050	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
50	0.849	1.047	1.299	1.676	2.009	2.403	2.678	2.937	3.261	3.496
60	0.848	1.045	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
80	0.846	1.043	1.292	1.664	1.990	2.374	2.639	2.887	3.195	3.416
100	0.845	1.042	1.290	1.660	1.984	2.364	2.626	2.871	3.174	3.390
150	0.844	1.040	1.287	1.655	1.976	2.351	2.609	2.849	3.145	3.357
Infinity	0.842	1.036	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.290

F Table for alpha=.05 Prob(F>entry)=0.05

df den.	df num. -->																		
	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	INF
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.277	9.117	9.014	8.941	8.887	8.845	8.812	8.786	8.745	8.703	8.660	8.639	8.617	8.594	8.572	8.549	8.526
4	7.709	6.944	6.591	6.388	6.256	6.163	6.094	6.041	5.999	5.964	5.912	5.858	5.803	5.774	5.746	5.717	5.688	5.658	5.628
5	6.608	5.786	5.410	5.192	5.050	4.950	4.876	4.818	4.773	4.735	4.678	4.619	4.558	4.527	4.496	4.464	4.431	4.399	4.365
6	5.987	5.143	4.757	4.534	4.387	4.284	4.207	4.147	4.099	4.060	4.000	3.938	3.874	3.842	3.808	3.774	3.740	3.705	3.669
7	5.591	4.737	4.347	4.120	3.972	3.866	3.787	3.726	3.677	3.637	3.575	3.511	3.445	3.411	3.376	3.340	3.304	3.267	3.230
8	5.318	4.459	4.066	3.838	3.688	3.581	3.501	3.438	3.388	3.347	3.284	3.218	3.150	3.115	3.079	3.043	3.005	2.967	2.928
9	5.117	4.257	3.863	3.633	3.482	3.374	3.293	3.230	3.179	3.137	3.073	3.006	2.937	2.901	2.864	2.826	2.787	2.748	2.707
10	4.965	4.103	3.708	3.478	3.326	3.217	3.136	3.072	3.020	2.978	2.913	2.845	2.774	2.737	2.700	2.661	2.621	2.580	2.538
11	4.844	3.982	3.587	3.357	3.204	3.095	3.012	2.948	2.896	2.854	2.788	2.719	2.646	2.609	2.571	2.531	2.490	2.448	2.405
12	4.747	3.885	3.490	3.259	3.106	2.996	2.913	2.849	2.796	2.753	2.687	2.617	2.544	2.506	2.466	2.426	2.384	2.341	2.296
13	4.667	3.806	3.411	3.179	3.025	2.915	2.832	2.767	2.714	2.671	2.604	2.533	2.459	2.420	2.380	2.339	2.297	2.252	2.206
14	4.600	3.739	3.344	3.112	2.958	2.848	2.764	2.699	2.646	2.602	2.534	2.463	2.388	2.349	2.308	2.266	2.223	2.178	2.131
15	4.543	3.682	3.287	3.056	2.901	2.791	2.707	2.641	2.588	2.544	2.475	2.403	2.328	2.288	2.247	2.204	2.160	2.114	2.066
16	4.494	3.634	3.239	3.007	2.852	2.741	2.657	2.591	2.538	2.494	2.425	2.352	2.276	2.235	2.194	2.151	2.106	2.059	2.010
17	4.451	3.592	3.197	2.965	2.810	2.699	2.614	2.548	2.494	2.450	2.381	2.308	2.230	2.190	2.148	2.104	2.058	2.011	1.960
18	4.414	3.555	3.160	2.928	2.773	2.661	2.577	2.510	2.456	2.412	2.342	2.269	2.191	2.150	2.107	2.063	2.017	1.968	1.917
19	4.381	3.522	3.127	2.895	2.740	2.628	2.544	2.477	2.423	2.378	2.308	2.234	2.156	2.114	2.071	2.026	1.980	1.930	1.878
20	4.351	3.493	3.098	2.866	2.711	2.599	2.514	2.447	2.393	2.348	2.278	2.203	2.124	2.083	2.039	1.994	1.946	1.896	1.843
21	4.325	3.467	3.073	2.840	2.685	2.573	2.488	2.421	2.366	2.321	2.250	2.176	2.096	2.054	2.010	1.965	1.917	1.866	1.812
22	4.301	3.443	3.049	2.817	2.661	2.549	2.464	2.397	2.342	2.297	2.226	2.151	2.071	2.028	1.984	1.938	1.889	1.838	1.783
23	4.279	3.422	3.028	2.796	2.640	2.528	2.442	2.375	2.320	2.275	2.204	2.128	2.048	2.005	1.961	1.914	1.865	1.813	1.757
24	4.260	3.403	3.009	2.776	2.621	2.508	2.423	2.355	2.300	2.255	2.183	2.108	2.027	1.984	1.939	1.892	1.842	1.790	1.733
25	4.242	3.385	2.991	2.759	2.603	2.490	2.405	2.337	2.282	2.237	2.165	2.089	2.008	1.964	1.919	1.872	1.822	1.768	1.711
26	4.225	3.369	2.975	2.743	2.587	2.474	2.388	2.321	2.266	2.220	2.148	2.072	1.990	1.946	1.901	1.853	1.803	1.749	1.691
27	4.210	3.354	2.960	2.728	2.572	2.459	2.373	2.305	2.250	2.204	2.132	2.056	1.974	1.930	1.884	1.836	1.785	1.731	1.672
28	4.196	3.340	2.947	2.714	2.558	2.445	2.359	2.291	2.236	2.190	2.118	2.041	1.959	1.915	1.869	1.820	1.769	1.714	1.654
29	4.183	3.328	2.934	2.701	2.545	2.432	2.346	2.278	2.223	2.177	2.105	2.028	1.945	1.901	1.854	1.806	1.754	1.698	1.638
30	4.171	3.316	2.922	2.690	2.534	2.421	2.334	2.266	2.211	2.165	2.092	2.015	1.932	1.887	1.841	1.792	1.740	1.684	1.622
40	4.085	3.232	2.839	2.606	2.450	2.336	2.249	2.180	2.124	2.077	2.004	1.925	1.839	1.793	1.744	1.693	1.637	1.577	1.509
60	4.001	3.150	2.758	2.525	2.368	2.254	2.167	2.097	2.040	1.993	1.917	1.836	1.748	1.700	1.649	1.594	1.534	1.467	1.389
120	3.920	3.072	2.680	2.447	2.290	2.175	2.087	2.016	1.959	1.911	1.834	1.751	1.659	1.608	1.554	1.495	1.429	1.352	1.254
inf	3.842	2.996	2.605	2.372	2.214	2.099	2.010	1.938	1.880	1.831	1.752	1.666	1.571	1.517	1.459	1.394	1.318	1.221	1.000